SIPTA

SIPTA Summer School
15–19 August 2022
University of Bristol

Imprecision

(not as a problem, but as part of the solution)

Gert de Cooman

Foundations Lab
for imprecise probabilities

GHENT UNIVERSITY

I have learnt from (talking to and working with) many …

Teddy Seidenfeld


Peter Walley


Glenn Shafer


Volodya Vovk


Philip Dawid


Marco Zaffalon


Enrique Miranda


Matthias Troffaes

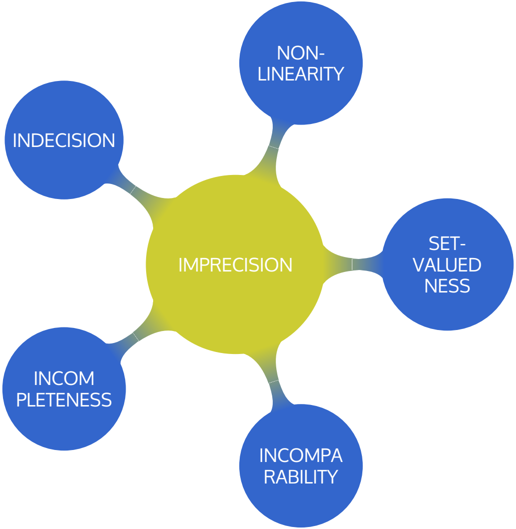

Erik Quaeghebeur


Jasper De Bock


Arthur Van Camp


Floris Persiau

Imprecision comes in many guises …



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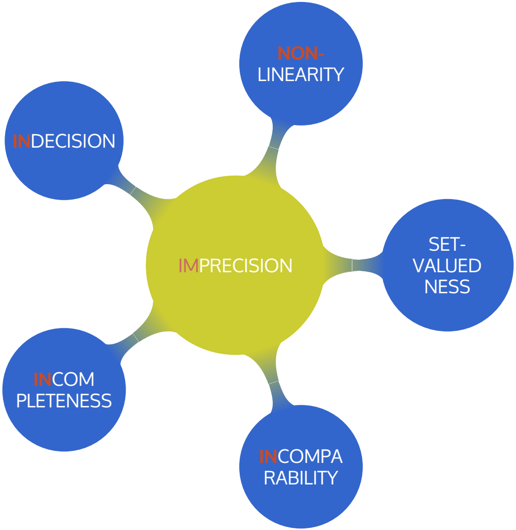
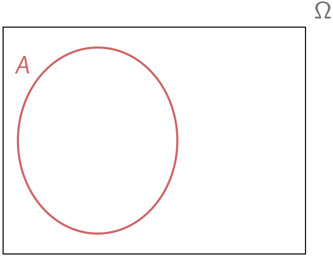
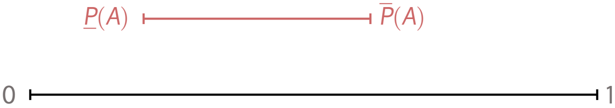


PROBABILITY INTERVALS

Probability intervals





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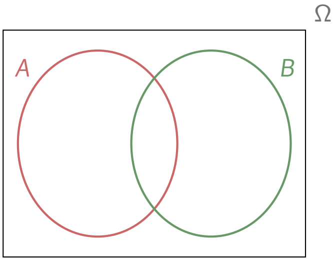

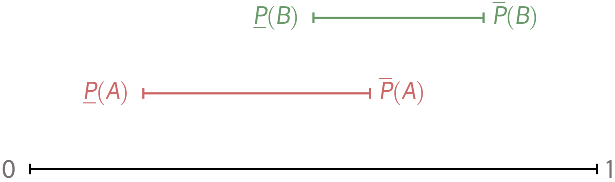

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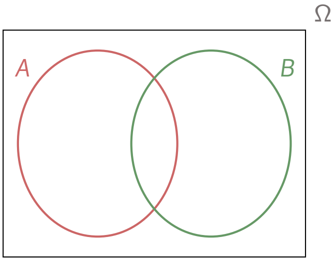
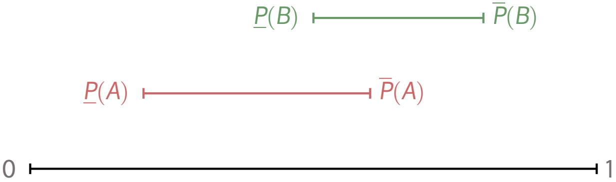
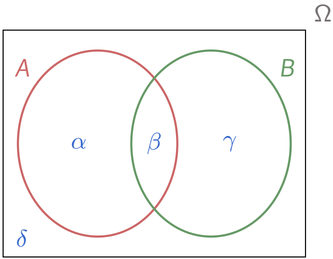




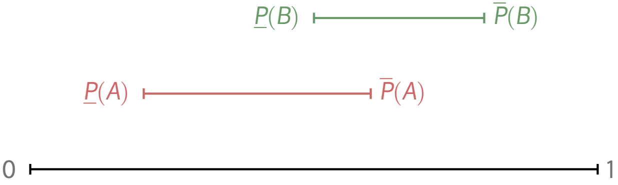
$\underline{P}(A \cap B) = ?$ $\qquad$ $\overline{P}(A \cap B) = ?$

Probability intervals



Maximise and minimise β under the constraints:

$$\alpha + \beta + \gamma + \delta = 1$$
$$\alpha, \beta, \gamma, \delta \geqslant 0$$
$$\underline{P}(A) \leqslant \alpha + \beta \leqslant \overline{P}(A)$$
$$\underline{P}(B) \leqslant \beta + \gamma \leqslant \overline{P}(B)$$



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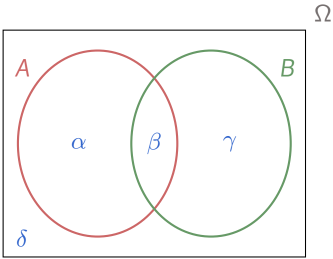

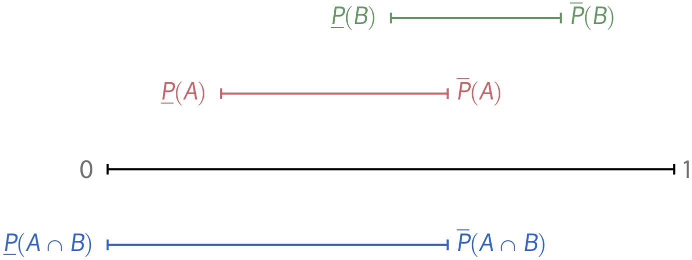

Maximise and minimise β under the constraints:

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$$\underline{P}(A) \leqslant \alpha + \beta \leqslant \overline{P}(A)$$
$$\underline{P}(B) \leqslant \beta + \gamma \leqslant \overline{P}(B)$$

$\underline{P}(A \cap B) = \max\{0, \underline{P}(A) + \underline{P}(B) - 1\}$ and $\overline{P}(A \cap B) = \min\{\overline{P}(A), \overline{P}(B)\}$

CREDAL SETS

Sets of probability measures

$\Downarrow$

closed and convex
set of probability measures $\mathcal{M}$ on Ω

$=$

CREDAL SET

Sets of probability measures

$\Downarrow$

closed and convex

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$=$

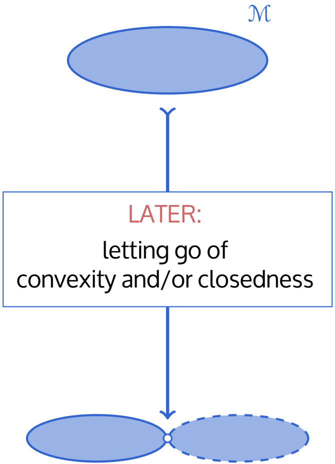
CREDAL SET

$\Downarrow$

lower and upper envelopes of $\mathcal{M}$:

$$\underline{P}(C) := \min\{P(C) \colon P \in \mathcal{M}\} \text{ and } \overline{P}(C) := \max\{P(C) \colon P \in \mathcal{M}\}$$

Sets of probability measures



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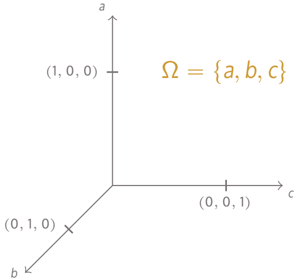
EXPECTATION INTERVALS

Expectation intervals



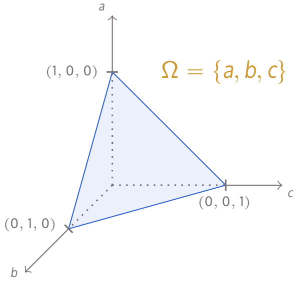
$$(1, 0, 0) \equiv \mathbb{I}_{\{a\}}$$

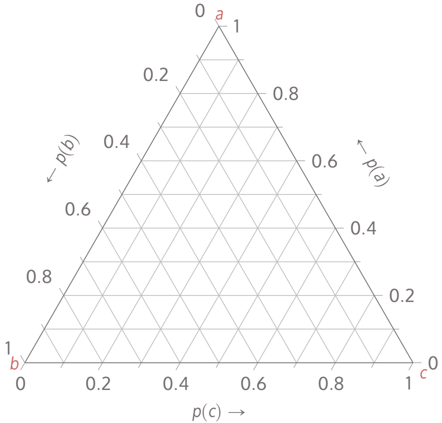
$$(0, 1, 0) \equiv \mathbb{I}_{\{b\}}$$

$$(0, 0, 1) \equiv \mathbb{I}_{\{c\}}$$

$$(\alpha, \beta, \gamma) \equiv \underbrace{\alpha\mathbb{I}_{\{a\}} + \beta\mathbb{I}_{\{b\}} + \gamma\mathbb{I}_{\{c\}}}_{\text{gamble } g\colon\, \Omega\to\mathbb{R}}$$

Expectation intervals





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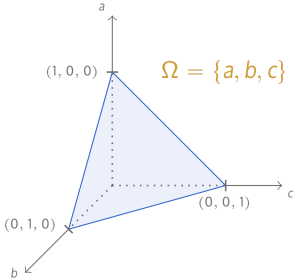

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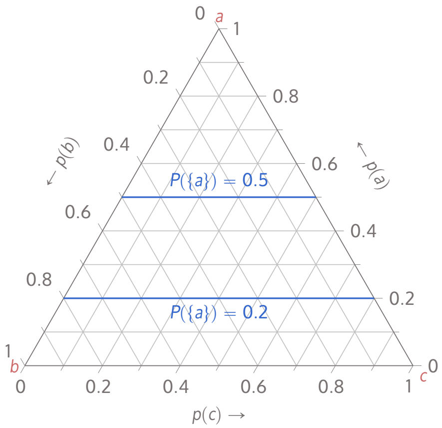

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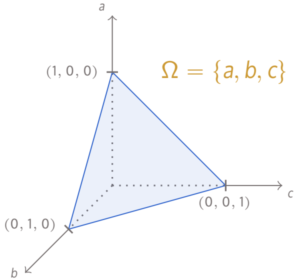

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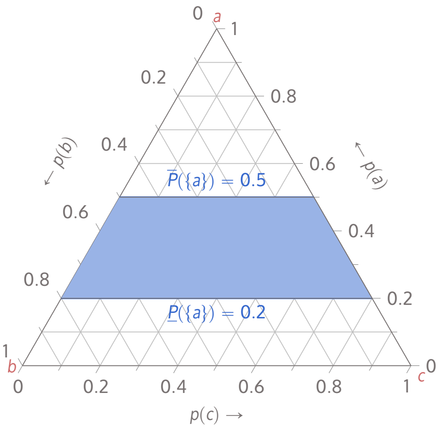

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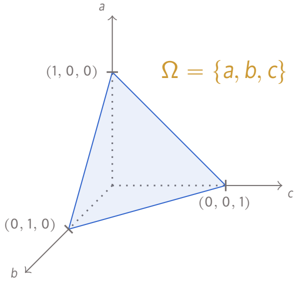

$\Omega = \{a, b, c\}$

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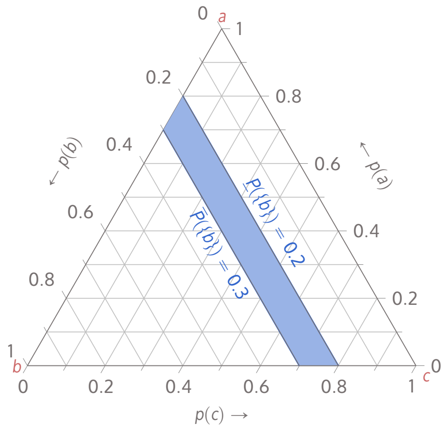

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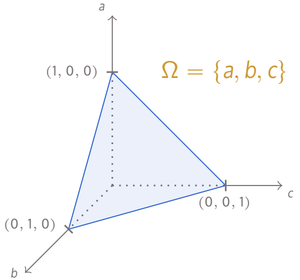

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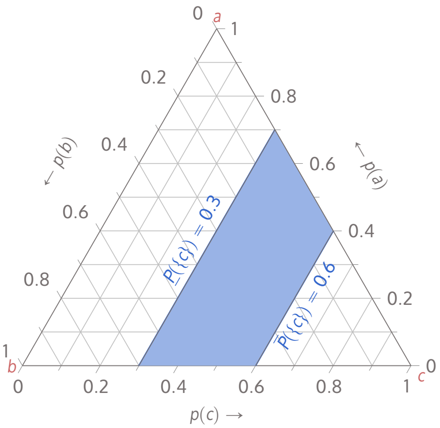

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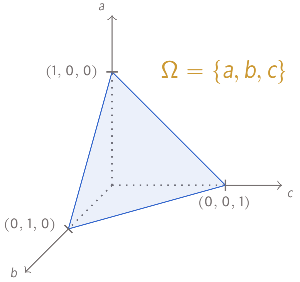

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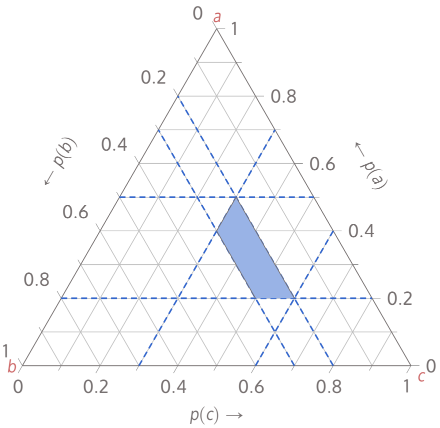

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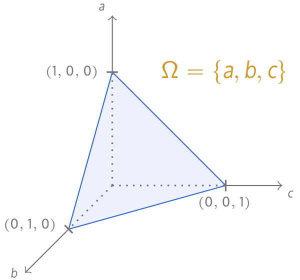

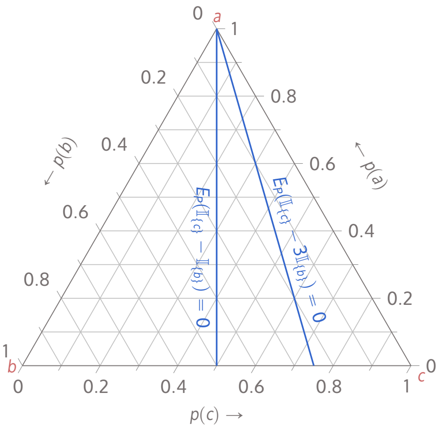

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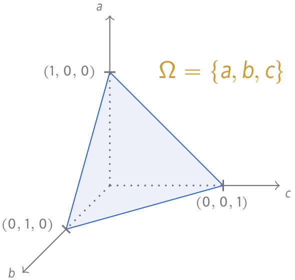

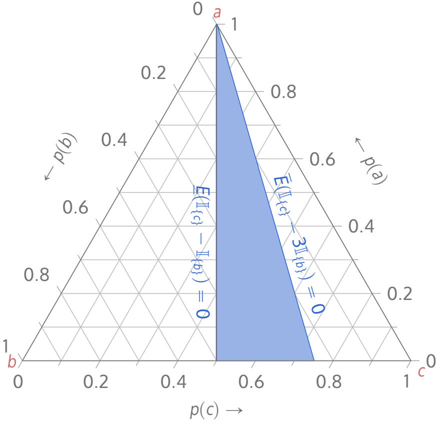

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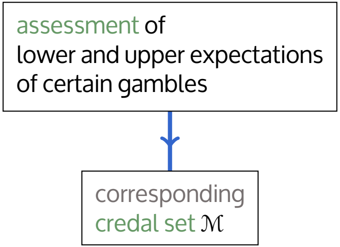
Expectation intervals

assessment of
lower and upper expectations
of certain gambles

Expectation intervals



Expectation intervals

assessment of lower and upper expectations of certain gambles

↓

corresponding credal set $\mathcal{M}$

↓

lower and upper envelopes

$$\underline{E}(g) := \min\{E_P(g) : P \in \mathcal{M}\}$$
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CONSERVATIVE INFERENCE

LOWER AND UPPER EXPECTATIONS

Lower and upper expectations

Lower expectation:

- $\inf f \leqslant \underline{E}(f)$
- $\underline{E}(f+g) \geqslant \underline{E}(f) + \underline{E}(g)$
- $\underline{E}(\lambda f) = \lambda \underline{E}(f)$ for $\lambda \geqslant 0$

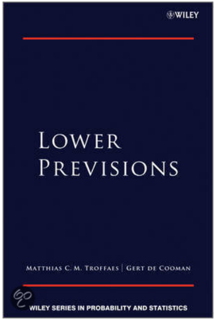
Conjugacy:

- $\overline{E}(f) = -\underline{E}(-f)$

Expectation:

- $\inf f \leqslant E(f)$
- $E(f+g) = E(f) + E(g)$
- $E(\lambda f) = \lambda E(f)$ for all λ

Lower and upper expectations



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Conjugacy:

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Expectation:

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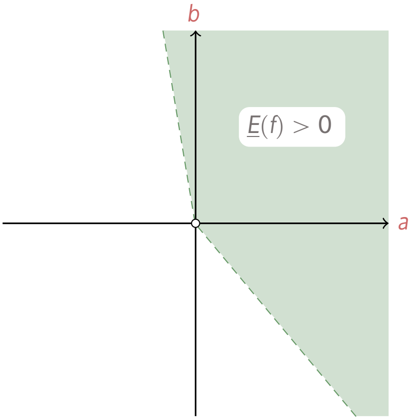
DECISION MAKING

Decision making

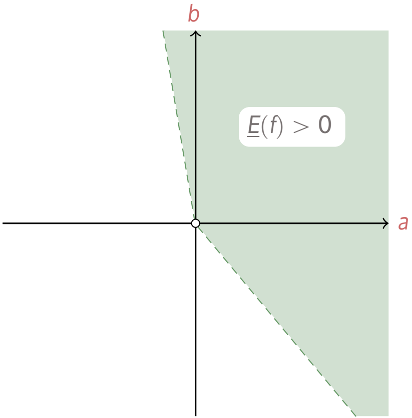
$f \succ 0 \Leftrightarrow \left(\underline{E}(f) > 0 \text{ or } f > 0\right)$



Decision making

$$f \succ 0 \Leftrightarrow \big(\underline{E}(f) > 0 \text{ or } f > 0\big)$$

$$g \succ f \Leftrightarrow g - f \succ 0$$



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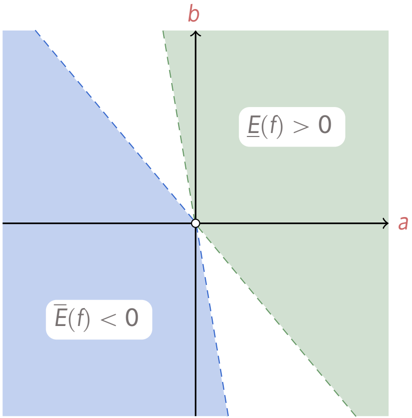
$$g \succ f \Leftrightarrow g - f \succ 0$$

$$0 \succ f \Leftrightarrow -f \succ 0$$
$$\Leftrightarrow \left(\overline{E}(f) < 0 \text{ or } f < 0\right)$$



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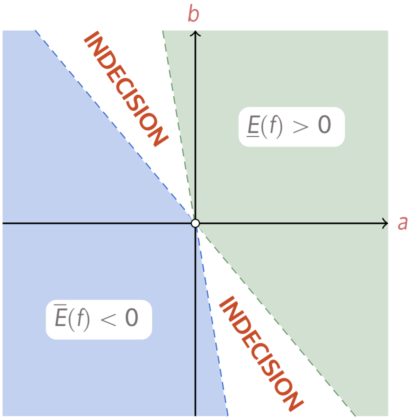

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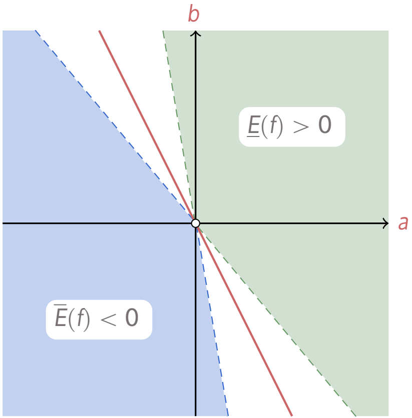
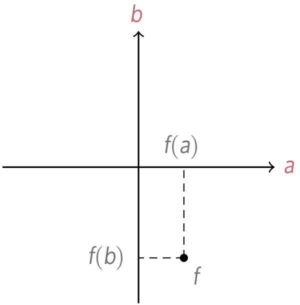


DESIRABLE GAMBLES

Binary choice: desirable gambles

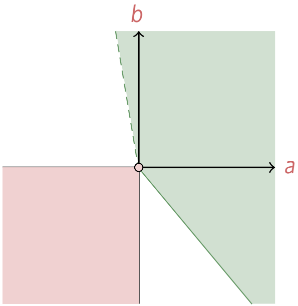


A gamble $f\colon \Omega \to \mathbb{R}$ is desirable if it is strictly preferred to the zero gamble 0—the status quo.

The logic of desirability is based on elementary statements

$$\vdash_D f \longrightarrow \text{'the gamble } f \text{ is desirable'}.$$

Binary choice: desirable gambles



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The logic of desirability is based on elementary statements

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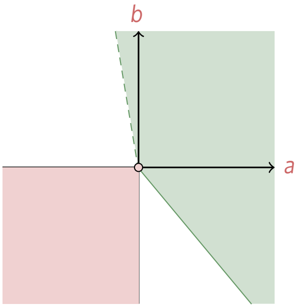
This logic is governed by the following axioms:

+ $\nvdash_D 0$
+ if $f > 0$ then $\vdash_D f$
+ if $(\vdash_D f$ and $\vdash_D g)$ then $\vdash_D (f + g)$
+ if $\vdash_D f$ then $\vdash_D (\lambda f)$ for all real $\lambda > 0$

Binary choice: desirable gambles



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The logic of desirability is based on elementary statements

$$\vdash_{\mathrm{D}} f \longrightarrow \text{'the gamble } f \text{ is desirable'}.$$

This logic is governed by the following axioms:

+ $\not\vdash_{\mathrm{D}} 0$
+ if $f > 0$ then $\vdash_{\mathrm{D}} f$
+ if $(\vdash_{\mathrm{D}} f$ and $\vdash_{\mathrm{D}} g)$ then $\vdash_{\mathrm{D}} (f + g)$
+ if $\vdash_{\mathrm{D}} f$ then $\vdash_{\mathrm{D}} (\lambda f)$ for all real $\lambda > 0$

CONSERVATIVE INFERENCE

Binary choice: desirable gambles

The logic behind desirable gambles underlies all of (finitary) probability theory.

$$\underline{E}(f) := \sup\{\alpha \in \mathbb{R} \colon \ \vdash_{\mathrm{D}} (f - \alpha)\ \}$$

Binary choice: desirable gambles

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$$\underline{E}(f) := \sup\{\alpha \in \mathbb{R} \colon \ \vdash_{\mathsf{D}} (f - \alpha)\}$$

Bayes's rule is part of this logic and is therefore deductive:

$$\underline{E}(f|A) := \sup\{\alpha \in \mathbb{R} \colon \ \vdash_{\mathsf{D}} (f - \alpha)\mathbb{I}_A\}$$

Binary choice: desirable gambles

ALLOWING FOR IMPRECISION LAYS BARE THE CONSERVATIVE INFERENCE MECHANISM BEHIND PROBABILISTIC REASONING

The logic behind desirable gambles underlies all of (finitary) probability theory.

$$\underline{E}(f) := \sup\{\alpha \in \mathbb{R} : \vdash_{\mathrm{D}} (f - \alpha)\}$$

Bayes's rule is part of this logic and is therefore deductive:

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CHOICE FUNCTIONS

Non-binary choice: desirable sets of gambles

RECALL: BINARY CHOICE

The logic of desirable gambles is governed by:

+ $\nvdash_{\mathrm{D}} 0$
+ if $f > 0$ then $\vdash_{\mathrm{D}} f$
+ if $\vdash_{\mathrm{D}} f, g$ then $\vdash_{\mathrm{D}} (f + g)$
+ if $\vdash_{\mathrm{D}} f$ then $\vdash_{\mathrm{D}} (\lambda f)$ for all real $\lambda > 0$

Non-binary choice: desirable sets of gambles

RECALL: BINARY CHOICE

The logic of desirable gambles is governed by:

- $\not\vdash_{\mathrm{D}} 0$
- if $f > 0$ then $\vdash_{\mathrm{D}} f$
- if $\vdash_{\mathrm{D}} f, g$ then $\vdash_{\mathrm{D}} (f + g)$
- if $\vdash_{\mathrm{D}} f$ then $\vdash_{\mathrm{D}} (\lambda f)$ for all real $\lambda > 0$

We can get to NON-BINARY CHOICE and the theory of choice functions by (essentially) adding an extra idea:

- if $\big(\vdash_{\mathrm{D}} f_1$ or $\dots$ or $\vdash_{\mathrm{D}} f_n\big)$ then $\vdash_{\mathrm{D}} \{f_1, \dots, f_n\}$

Non-binary choice: desirable sets of gambles

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PROPOSITIONAL LOGIC ONLY WITH STATEMENTS:
'f is desirable'

Non-binary choice: desirable sets of gambles

REPRESENTATION RESULTS:

Levi's E-admissibility but with

- coherent set of desirable gambles instead of probability measure
- sets not necessarily closed nor convex
- extra axioms add extra structure

RECALL: BINARY CHOICE

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PROPOSITIONAL LOGIC ONLY WITH STATEMENTS: 'f is desirable'

Non-binary choice: desirable sets of gambles

ALLOWING FOR IMPRECISION LAYS BARE THE LINK BETWEEN PROBABILISTIC REASONING AND CHOICE THEORY, AND EXTENDS IT SIGNIFICANTLY

RECALL: BINARY CHOICE

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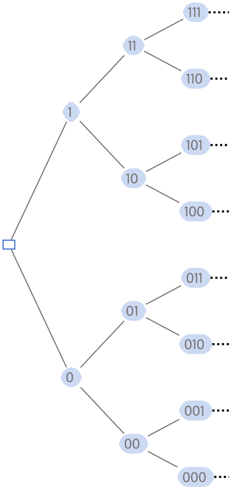
PROPOSITIONAL LOGIC ONLY WITH STATEMENTS: 'f is desirable'

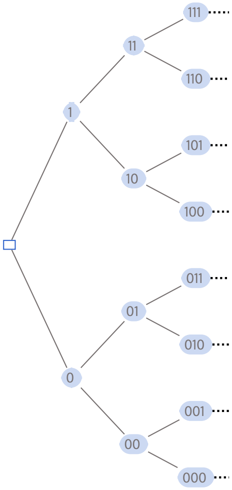
IMPRECISE STOCHASTIC PROCESSES

Imprecision in stochastic processes



$X_1, X_2, X_3, \ldots$

Imprecision in stochastic processes

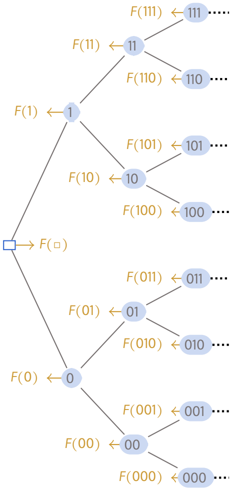


$X_1, X_2, X_3, \ldots$

Situations $s \in \mathbb{S}$ are the nodes in the event tree:
finite strings of states

Paths $\omega \in \Omega$ are the leaves in the event tree:
infinite strings of states

Imprecision in stochastic processes



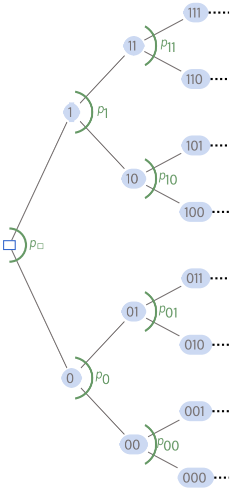
$X_1, X_2, X_3, \ldots$

Situations $s \in \mathbb{S}$ are the nodes in the event tree: finite strings of states

Paths $\omega \in \Omega$ are the leaves in the event tree: infinite strings of states

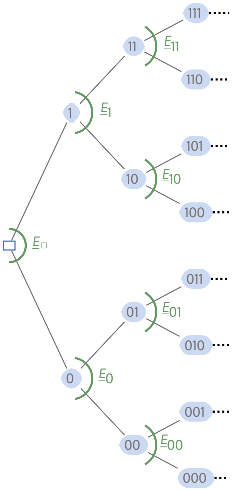
A process $F\colon \mathbb{S} \to \mathbb{R}$ attaches a real number $F(s)$ to every situation s.

Imprecision in stochastic processes



A precise probability tree attaches a local mass function p_s to every situation s.

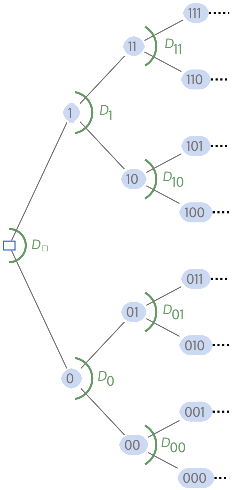
Imprecision in stochastic processes



A precise probability tree attaches a local mass function p_s to every situation s.

An imprecise probability tree attaches a local lower expectation $\underline{E}_s$ to every situation s.

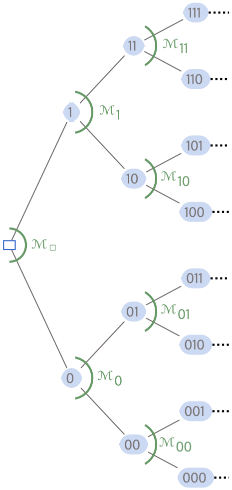
Imprecision in stochastic processes



A precise probability tree attaches a local mass function p_s to every situation s.

An imprecise probability tree attaches a local set of desirable gambles D_s to every situation s.

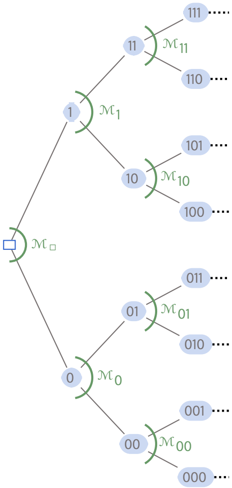
Imprecision in stochastic processes



A precise probability tree attaches a local mass function p_s to every situation s.

An imprecise probability tree attaches a local credal set $\mathcal{M}_s$ to every situation s.

Imprecision in stochastic processes



A precise probability tree attaches a local mass function p_s to every situation s.

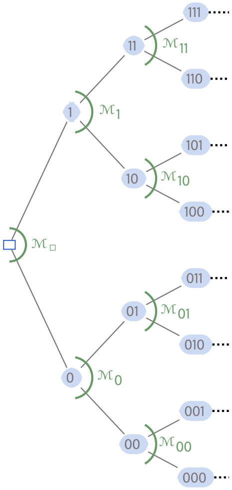
An imprecise probability tree attaches a local credal set $\mathcal{M}_s$ to every situation s.

An imprecise probability tree is equivalent to a set of precise probability trees.

An imprecise probability tree is equivalent to a convex closed set of special processes, called supermartingales.

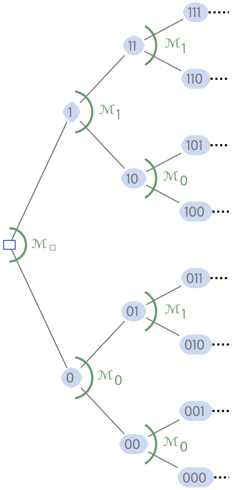
IMPRECISE
MARKOV CHAINS

Imprecision in Markov chains



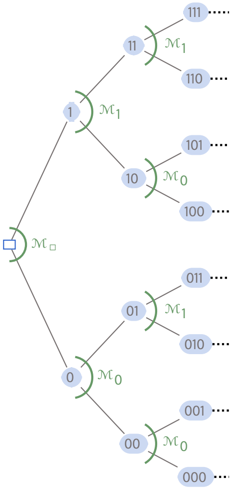
$$\mathcal{M}_{(x_1,\dots,x_n)} = \mathcal{M}_{x_n}$$

Imprecision in Markov chains



$$\mathcal{M}_{(x_1,\dots,x_n)} = \mathcal{M}_{x_n}$$

Imprecision in Markov chains

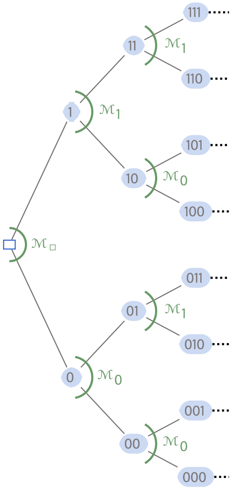


Due to the Markov condition

$$\mathcal{M}_{(x_1,\dots,x_n)} = \mathcal{M}_{x_n},$$

many inferences in imprecise Markov chains become polynomial in complexity, no longer exponential.

Imprecision in Markov chains

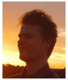


Due to the Markov condition

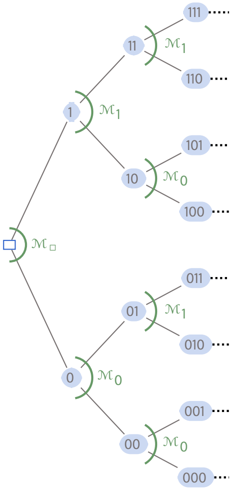
$$\mathcal{M}_{(x_1,\dots,x_n)} = \mathcal{M}_{x_n},$$

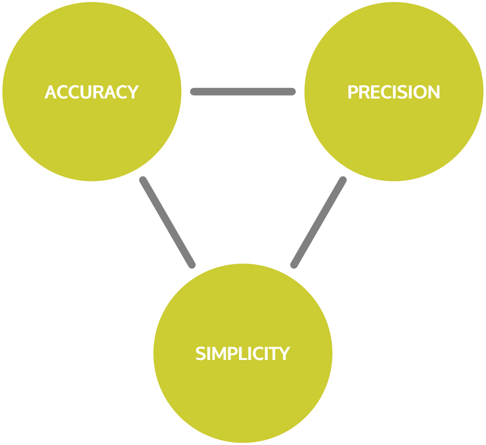
many inferences in imprecise Markov chains become polynomial in complexity, no longer exponential.

By allowing for imprecision, we can efficiently calculate conservative bounds on the behaviour of precise stochastic processes that are not Markov.

 and    $\Rightarrow$ LUMPING

Imprecision in Markov chains



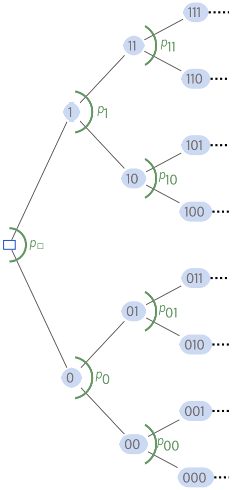


ALGORITHMIC RANDOMNESS

Imprecision in algorithmic randomness

0110011010101111100101110001110101010011011011100011...

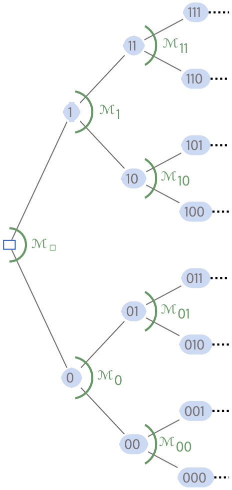
Imprecision in algorithmic randomness



0110011010101111100101110001110101010011011011100011...

random for a precise probability tree →randomness tests →supermartingales

Imprecision in algorithmic randomness

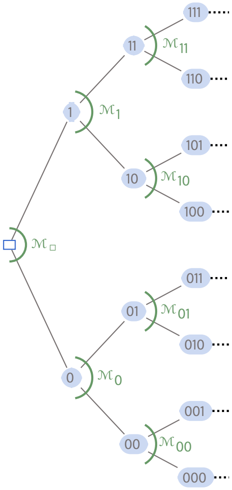


0110011010101111100101110001110101010011011011100011...

random for a precise probability tree →randomness tests
→supermartingales

random for an imprecise probability tree →randomness tests
→supermartingales

Imprecision in algorithmic randomness

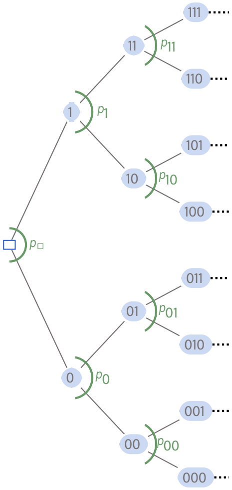


random for more precise probability tree

$\Downarrow$

random for less precise probability tree

Imprecision in algorithmic randomness

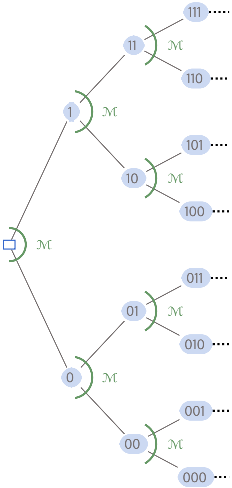


random for more precise probability tree

⇓ but not ⇑

random for less precise probability tree

Imprecision in algorithmic randomness



random for more precise probability tree

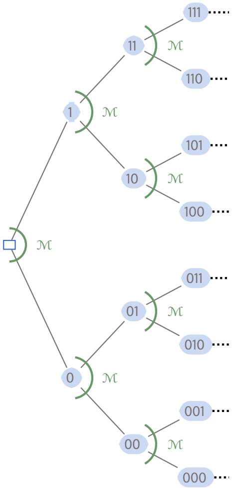
$\Downarrow$ but not $\Uparrow$

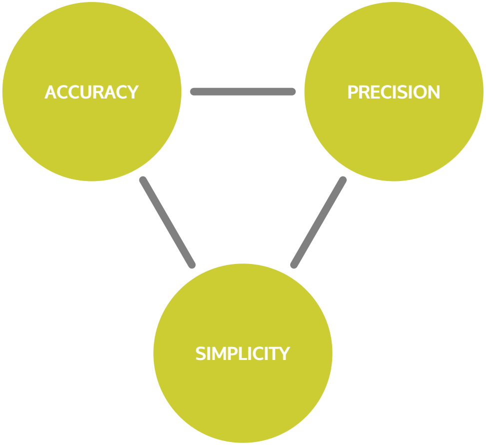
random for less precise probability tree

EXAMPLE:

NONSTATIONARY PRECISE $\Rightarrow$ STATIONARY IMPRECISE

Imprecision in algorithmic randomness





IN SUMMARY

In summary …

IMPRECISION IN PROBABILITY THEORY ALLOWS US AND HELPS US TO

+ deal honestly and systematically with indecision and incompleteness
+ see the precise special case in a much wider, structured mathematical perspective and context
+ identify and use the logic and conservative inference mechanisms behind probabilistic reasoning
+ provide a natural link between measure-theoretic and game-theoretic probability
+ look at and use simpler and more conservative models that are computationally more tractable

THE END – FOR NOW