Session 1: Background on IP and de Finetti's *Coherence*

Partition circumstances with a finite set of

***pairwise exclusive* and *mutually exhaustive* situations.**

A partition with n-states {$state_1$, $state_2$, ….., $state_n$} is written as:

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}.$$

Suppose that YOU, the decision maker, can compare two acts, state by state, according to the desirability of their *outcomes*, o_{ij}.

	ω_1	ω_2	…	ω_k	…	ω_n
Act_1	o_{11}	o_{12}	…	o_{1k}	…	o_{1n}
Act_2	o_{21}	o_{22}	…	o_{2k}	…	o_{2n}

Strict dominance

If YOU judge each outcome o_{2j} is strictly preferable to the outcome o_{1j} ($j = 1,\ldots, n$),

then you strictly prefer Act_2 over Act_1 in a pairwise choice between them.

And Act_1 is inadmissible in any choice problem where Act_2 is available.

Example 1: Suppose that YOU prefer more money to less in each state. Consider the binary state decision problem where the payoffs are:

	ω_1	ω_2
Act_1	£*300*	£*100*
Act_2	£*400*	£*200*

So, Act_2 strictly dominates Act_1.

Might it be reasonable, nonetheless, to prefer Act_1 over Act_2 ??

For instance, what if Act_i brings about state ω_i? There is act/state dependence.
Suppose that the ω_i are options for a second decision maker who learns YOUR choice before deciding herself? What do you choose then?

- **Exercise 1: FILL IN THE DETAILS.**

This is an instance of what is called in the insurance business *Moral Hazard*.

WATCH OUT FOR MORAL HAZARDS!!

Three Variations on Dominance

we admit (countably) infinite partitions, $\Omega = \{\omega_1, \omega_2, \ldots\}$

	ω_1	ω_2	…	ω_k	…
Act_1	o_{11}	o_{12}	…	o_{1k}	…
Act_2	o_{21}	o_{22}	…	o_{2k}	…

YOU strictly prefer Act_2 over Act_1 in a pairwise choice between them, if

Uniform Dominance: There exists reward o^*, strictly preferred to status quo.
Each o_{2j} is strictly preferred to the composite outcome o_{1j} "+" o^*.

Simple Dominance:
Each o_{2j} is strictly preferred to o_{1j}.

Weak Dominance: Each o_{2j} is weakly preferred to o_{1j}, and for some j is strictly preferred.

- **If Act_2 uniformly dominates Act_1, then it simply dominates.**
- **And if it simply dominates, then it weakly dominates.**

Example 2 (Uniform Dominance): De Finetti's theory of *coherent* (2-sided) *fair prices* for buying/selling random variables – a 2-person, sequential, 0-sum game.

Consider a partition $\Omega = \{\omega_1, \omega_2, \ldots\}$
a field of sets $\mathcal{B}$ over Ω,
and a set $\chi = \{X_1, X_2, \ldots\}$ of (bounded) real-valued random variables on Ω
$X_i: \Omega \rightarrow \Re$ is a $\mathcal{B}$-measurable (bounded) function.

Bookie **(Player 1 – the merchant). If the *Bookie* chooses to play, rather than to *Abstain*, she/he is obliged to announce a *fair-price Price*(X_i) = q_i for each element of χ.**
If the game is played,
Gambler **(Player 2 – the customer) is allowed to make finitely many non-trivial contracts:**

When *Gambler* chooses the real-quantity γ_i for X_i, that fixes a <u>contract</u> where, in state ω, *Gambler* pays to the *Bookie* the amount

$$\gamma_i\, [\, X_i(\omega) - q_i\,].$$

When $\gamma_i > 0$, the *Bookie* buys γ_i-many units of X_i from the *Gambler* at the price q_i.

When $\gamma_i < 0$, the *Bookie* sells $|\gamma_i|$-many units of X_i to the *Gambler* at the price q_i.

The net payment from multiple contracts is the sum of the individual contracts.

Defn:. A *Bookie*'s set of fair-prices $\{q_i\}$ are *incoherent* if
the *Gambler* has a strategy (a set $\{\gamma^*_i\}$) that produces a net (uniformly) negative payoff to the *Bookie* for each state $\omega \in \Omega$: The *Bookie* faces a uniform sure-loss.

With an incoherent set of fair-prices, when the *Gambler* uses the strategy $\{\gamma^*_i\}$, the *Bookie* suffer a uniform sure-loss ($o_{ij} < e^* < 0$) compared with *Abstaining*.

	ω_1	ω_2	…	ω_k	…
Incoherent pricing	o_{11}	o_{12}	…	o_{1k}	…
Abstain from playing	*0*	*0*	…	*0*	…

de Finetti's *Coherence Theorem for (2-sided) Fair Prices*.

A set of fair-prices {q_i} is <u>coherent</u> ***if and only if***

There is a (finitely additive) probability *P* with each price $q_i = E_P(X_i)$

For a coherent strategy, each price q_i is the *P*-Expected value of X_i.

Let F be a $\mathcal{B}$-measurable event – a subset of Ω that belongs to $\mathcal{B}$.

Identify event F with its indicator variable:

$$F(\omega) = 1 \text{ if } \omega \in F$$

$$F(\omega) = 0 \text{ if } \omega \notin F$$

- **The *Bookie*'s pricing an event F in this game amounts to offering bets on/against F at the rates q_F : $(1-q_F)$.**
- **The *Gambler's* strategy γ_F determines the stake $|\gamma_F|$ in the winner-take-all bet, and who is on which side of the bet, depending on whether γ_F is positive or negative.**

From the *Bookie*'s perspective, to be coherent,

$$q_F = \mathrm{E}_P(F) = P(F).$$

Thus, each coherent strategy for pricing bets on events is to announce probabilities for these events, using a common (finitely additive) probability, P.

Exercise 2:

Suppose that with respect to a binary partition $\{E_1, E_2 (= E_1^c)\}$, the *Bookie* posts fair odds of $\{q_1 = .4$ and $q_2 = .7\}$.
Suppose the *Gambler* has a total budget of £10 with which to wager and the *Bookie*'s has a large budget, at least £1,000.

- **Show these are *incoherent* prices by giving a strategy for the *Gambler* that produces a sure-loss for the *Bookie* with bets that the *Gambler* can cover.**

- **Suppose the *Gambler* is a decision maker who subscribes to the principle of *Maximizing Subjective Expected Utility* – and she has a <u>linear</u> utility for money in the range of bets that are feasible for her (and the *Bookie*'s) budget:**

  U(£x) = x for outcomes in the range of bets she can afford.

  Suppose, also, the *Gambler* has a personal probability, $P_G(E_1) = 0.5$.

→ **What strategies maximize the *Gambler*'s Expected Utility?**

<u>Note</u>: Avoiding sure loss does *not* commit the decision maker to maximizing sure gain.

De Finetti's 2-sided ("fair") pricing, incomplete elicitation, and "weak-IP"

Let the partition Ω = {1, 2, 3, 4, 5, 6} formed by the outcome of rolling a six-sided die.

Let $\mathcal{E} = \{E_1, \ldots E_n\}$ be the events that the *Bookie* has to price.

For choosing a strategy $\{\gamma_i\}$, in addition to using the basic rule, '*Buy low and sell high*,' the *Gambler* considers what is the set of events for which the *Bookie* is committed to having well-defined fair odds.

- ***Q*: <u>What is the *closure* of the set of events for which the *Bookie* has fair odds</u>?**

Let *A* and *B* be disjoint events, $A \cap B = \phi$, and let $C = A \cup B$.

If the *Bookie* has posted fair odds q_A and q_B respectively on *A* and *B*, use the rules of the game so that the *Gambler* constructs a bet on *C* at the fair odds $q_C = q_A + q_B$.

<u>Note well that *C* may not belong to the set $\mathcal{E}$.</u>

If the *Bookie* has posted fair odds q_A and q_C respectively on *A* and *C*, show how the *Gambler* constructs a bet on *B* at the fair odds $q_B = q_C - q_A$.

Again, note well that *B* may not belong to the set $\mathcal{E}$.

- **De Finetti's *Fundamental Theorem* applied to sets of events.**

Suppose <u>coherent</u> 2-sided prices are given for each event *E* in a set $\mathcal{E}$ defined with respect to some basic partition $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n, \ldots\}$.

So, by de Finetti's Theorem, these coherent prices are probabilities.

Let $\mathcal{E}^*$ be events for which the rules of the game compel well defined prices.

Let *F* be another event defined on Ω but not necessarily in $\mathcal{E}$.

- **What are the *Bookie*'s options for adding a coherent (2-sided) price for *F*?**

Define: $\underline{\mathrm{F}} = \{E \in \mathcal{E}^*: E \subseteq F\}$

$\overline{\mathrm{F}} = \{E \in \mathcal{E}^*: F \subseteq E\}$

Let $\underline{P}(F) = \sup_{E \in \underline{\mathrm{F}}} q(E)$ **and** $\overline{P}(F) = \inf_{E \in \overline{\mathrm{F}}} q(E)$

- **Then, the price for *F* that remains coherent with those already assigned to events in $\mathcal{E}$ is any value from $\underline{P}(F)$ to $\overline{P}(F)$: $[\underline{P}(F), \overline{P}(F)]$**
- **Outside this closed interval, adding a price for *F* is incoherent with the other prices already given.**

***Note*: de Finetti's coherence criterion does not require the rational agent to identify betting odds beyond those for which the *Fundamental Theorem* constrains them.**

Specifically, the rational agent is *not* required by *coherence* to have probabilities defined on an algebra of events, let alone on a power-set of events.

It is sufficient to have probabilities defined *as-needed* for the arbitrary set $\mathcal{E}$, as might arise in a particular decision problem.

Exercise 3:

Ω = {1, 2, 3, 4, 5, 6} the outcome of rolling an ordinary die, as before.

$\mathcal{E}$ is the set of these four events $\mathcal{E}$ = { {1}, {3,6}, {1,2,3}, {1,2,4} }

Suppose the *Bookie* posts fair odds for these four events that agree with the judgment that the die is "fair."

P({1}) = 1/6; P({3,6}) = 1/3; P({1,2,3}) = P({1,2,4}) = 1/2.

The *Fundamental Theorem* identifies those events, and the values for which precise betting odds are required by coherence.

- ***Which events have coherent betting odds fixed by $\mathcal{E}$?***

Hint*: *Show that only 12 pairs of complementary events have definite odds!

Additional Notes this Exercise

The set of events with coherent betting odds fixed by events in $\mathcal{E}$ does *not* form an algebra. Only 24 of 64 events have precise previsions.

The pair of the sure-event and empty set.
Two singleton events and their five-atom complements
Five doubleton events and their four-atom complements
Four three-atom events and their three-atom complements

For instance, by the Fundamental Theorem,

$$\underline{P}(\{6\}) = 0 < \overline{P}(\{6\}) = 1/3;$$

likewise $\underline{P}(\{4\}) = 0 < \overline{P}(\{4\}) = 1/3;$

however, $P(\{4,6\}) = 1/3.$

- **Moreover, the smallest algebra containing these 4 events is the power set of all 64 events on Ω.**

But why do I qualify this result as "weak-IP" ?

***Subjective Expected Utility* thesis: A decision maker chooses as-if she/he has a personal probability P(•) over states of uncertainty, and a cardinal utility U(•) over outcomes, and maximizes subjective expected utility.**

Act_1 is dispreferred to Act_2 *if and only if* $\sum_j P(\omega_j)U(o_{1j}) \leq \sum_j P(\omega_j)U(o_{2j})$

Note: When acts and states are probabilistically independent, i.e.,

whenever $P(\omega_j) = P(\omega_j \mid Act_i)$ $\quad i = 1, \ldots m \quad j = 1, \ldots, n$

then *strict dominance* is a valid decision rule.

That is, when there is no moral hazard, and Act_2 strictly dominates Act_1,

then the Subjective Expected Utility of Act_2 is greater than of Act_1.

De Finetti's coherence argument requires that, in order to avoid a sure loss, the *Bookie* behaves as if maximizing expected value (where payoff = utility) for some personal probability over the states, and where there is no moral hazard in betting.

- **<u>NOTE</u>: We'll see that de Finetti's coherence criterion avoids concerns with moral hazard!**

The coherent 2-sided prices are *fair* because each contract has expected value 0.

Decision making for SEU theory rests primarily on two axioms:

<u>Axiom 1</u>: Rational choice is determined by a binary preference relation that satisfies the requirements of a weak-order: transitivity and completeness.

<u>Axiom 2</u>: Preference satisfies an *Independence* or *Cancellation* rule with respect to probability mixtures:

Act A is dispreferred to Act B *if and only if*

xAct A ⊕ (1-x) Act C is dispreferred to xAct B ⊕ (1-x) Act C

Other axioms are needed to ensure that the SEU representation uses <u>real-valued</u> probabilities and utilities, and that utility for outcomes are state-independent.

Ellsberg's (1961) Paradox for SEU theory.

We use only 2 rewards, £0 and £1,000 in the following decision problems.

***Background*: There is an urn containing 90 balls, one of which will be drawn at random, i.e., the probability is 1/90 of drawing a particular ball from the urn.**

- **30 of the 90 balls are colored RED.**
- **Of the remaining 60 each is either GREEN or BLUE, with no restrictions.**

From de Finetti's perspective, the problem stipulates precise (coherent) prices for the events Red and its complement (Green or Blue): P(Red) = 1/3, P(Green or Blue) = 2/3.

But $[\underline{P}(\text{Green}), \overline{P}(\text{Green})] = [0, 2/3] = [\underline{P}(\text{Blue}), \overline{P}(\text{Blue})]$

Evaluate each of two pairs of options.

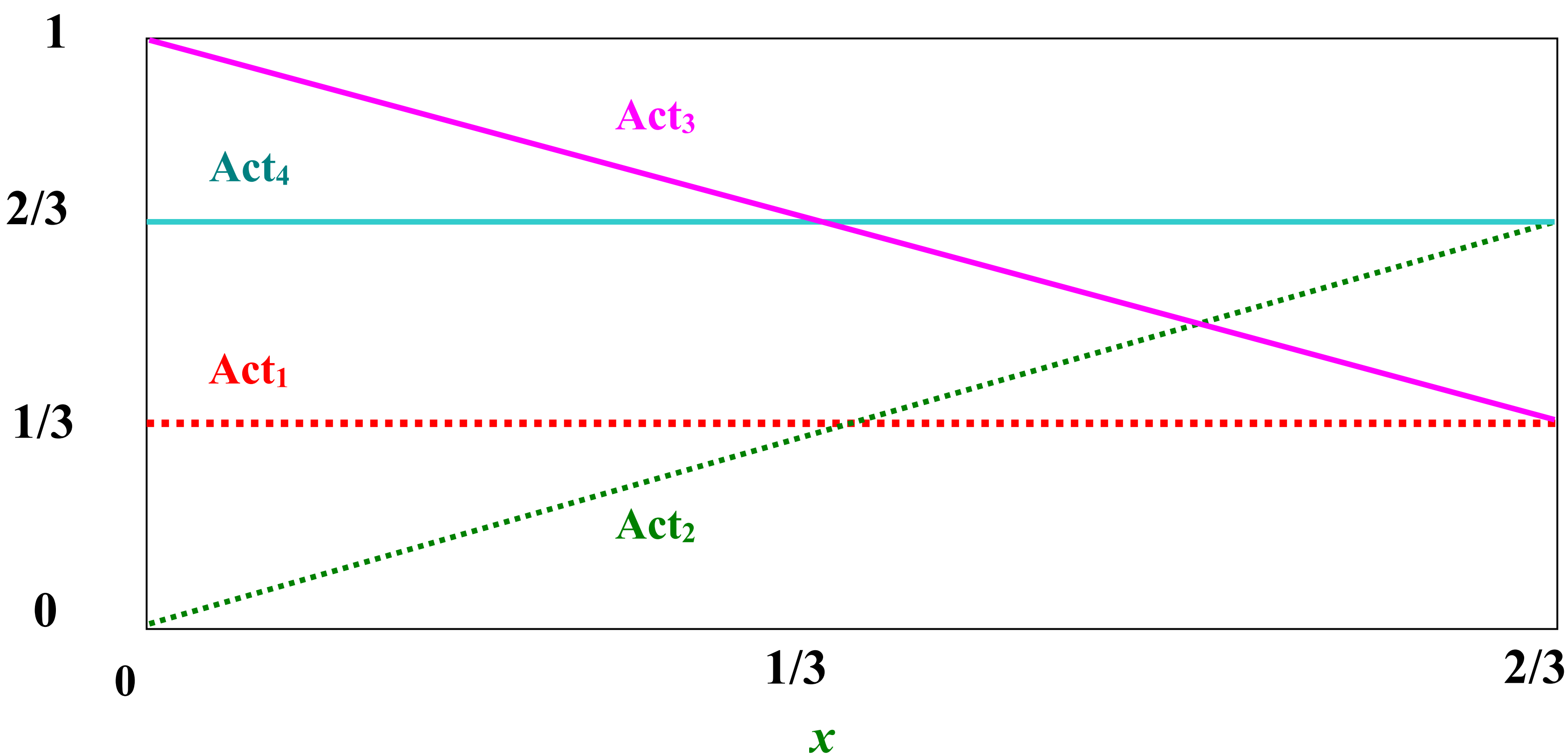
Act_1: Receive £1,000 if the ball drawn is RED, £0 otherwise.

Act_2: Receive £1,000 if the ball drawn is GREEN, £0 otherwise.

Act_3: Receive £1000 if the ball drawn is RED or BLUE, £0 if GREEN.

Act_4: Receive £1000 if the ball drawn is GREEN or BLUE, £0 if RED.

Consider the following graph of the expected utilities of these four acts as a function of x = personal probability of GREEN.



Note that, in terms of subjective expected utilities over different values of the unknown proportion of GREEN balls ($0 \leq x \leq 2/3$):

Act$_1$ maximizes minimum expected value (1/3) compared with Act$_2$ (0).

and Act$_4$ maximizes minimum expected value (2/3) compared with Act$_3$ (1/3).

- **Principle of *MaxiMin* in decisions under uncertainty, where not all prices are precise**:

Among the (undominated) options available,

choose that act whose minimum <expected> value is maximum.

So, in the Ellsberg problem, by *MaxiMin*, the decision maker chooses

Act_1 over Act_2 and Act_4 over Act_3.

But there are no coherent prices for the events {Red, Green, Blue } that agree with these choices, as represented by strict preferences.

- **So, de Finetti's *Fundamental Theorem*, though it offers an interval-valued interpretation of the agent's initial credal position about, e.g., Green, that sense of IP fails to underwrite the concern over uncertainty about Green versus Red that is revealed by the MaxiMin choices.**

  Nonetheless, de Finetti's theory has served as a fruitful basis for IP theory.

Basic IP and coherent 1-sided pricing

Modify de Finetti's game so that the *Bookie* is *allowed* to distinguish

a (supremum) buying price $\underline{q}(X)$

and a (infimum) selling price $\overline{q}(X)$.

***Aside*: Here, I am ignoring the subtle, but important issues relating to the endpoints of these price intervals!**

When $\underline{q}(X) = \overline{q}(X)$, the *Bookie* has a 2-sided "fair" price for *X*, in de Finetti's sense.

Rather than requiring one (fair) 2-sided price, $q(X)$, the *Bookie* is required to post a pair $\{\underline{q}(X), \overline{q}(X)\}$ that constrain the *Gambler*'s strategies.

When $\gamma_X > 0$, then the contract uses the *Bookie*'s buying price, $\underline{q}(X)$,

$$\gamma_x \, [\, X_i(\omega) - \underline{q}(X) \,].$$

When $\gamma_X < 0$, then the contract uses the *Bookie*'s selling price, $\overline{q}(X)$,

$$\gamma_x \, [\, X_i(\omega) - \overline{q}(X)].$$

IP version of de Finetti's *Coherence Theorem* for pairs of 1-sided prices.

The sequential, 2-person, 0-sum game is played, as before, with the added constraint that the *Bookie*'s prices are one sided.

***Note*: Assume that a constant variable $X_c(\omega) = c$ has a 2-sided price, $q(X_c) = c$.**

- ***Coherence* means the same respect for (uniform) Dominance.**

A set of 1-sided prices $\{\{\underline{q}(X), \overline{q}(X)\}: X \text{ in in } \mathcal{X}\}$ is <u>coherent</u>

if and only if

there is a convex set $\mathcal{P} = \{P\}$ of (finitely additive) probabilities on the measurable space $< \Omega, \mathcal{B} >$, such that:

$$\underline{q}(X) = \mathit{infimum}_{P\in\mathcal{P}}\, E_P(X)$$

and

$$\overline{q}(X) = \mathit{supremum}_{P\in\mathcal{P}}\, E_P(X).$$

***Aside*: Again, I am ignoring the technical issues relating to the endpoints of these price intervals, which ignores the technical issues relating to the boundaries of the set $\mathcal{P}$.**