SIPTA

SIPTA Summer School
15–19 August 2022
University of Bristol

Desirability

– the basics of inference under uncertainty

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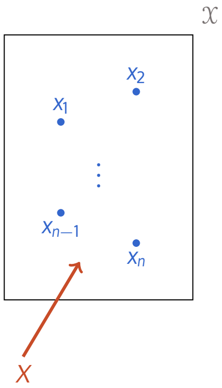
GHENT UNIVERSITY

Foundations Lab
for imprecise probabilities

BASIC SETUP

Uncertainty about a variable



You are uncertain about the value of a variable X in a set $\mathcal{X}$.

- You = a subject aiming to be rational
- possibility space $\mathcal{X} = \{x_1, \ldots, x_n\}$ is finite!
- notation:

  lowercase x = a possible value, or the actual value
  uppercase X = the unknown value

Gambles on a variable

A gamble g (on the variable X) is an uncertain reward whose value depends on the value of X in $\mathcal{X}$.

Rewards are expressed in units of some linear utility scale.

$$g\colon \mathcal{X} \to \mathbb{R}$$

We also denote this gamble by $g(X)$.

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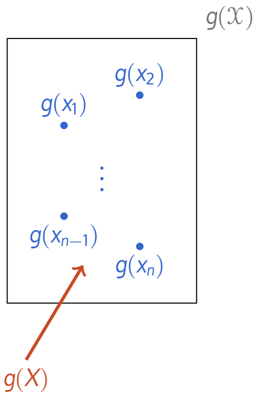

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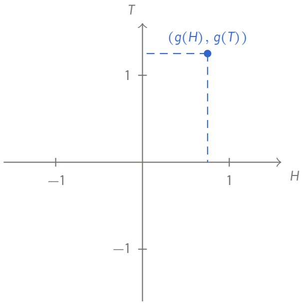
The gamble g is completely determined by the n-tuple

$$\big(g(x_1), \dots, g(x_n)\big) \in \mathbb{R}^n.$$

Gambles on a variable

$$\mathcal{X} = \{H, T\}$$



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The set of all gambles $\mathcal{L}(\mathcal{X})$ is a linear space that is isomorphic to $\mathbb{R}^n$.

PREFERENCE AND DESIRABILITY

Strict preference orderings

Your beliefs about what value X assumes in $\mathcal{X}$ may lead you to (strictly) prefer one uncertain reward $g(X)$ over another $h(X)$:

$$g \succ h.$$

Strict preference orderings

Weak strict ordering:

W1. $g \succ g$ for no $g \in \mathcal{L}(\mathcal{X})$;

W2. if $g_1 \succ g_2$ and $g_2 \succ g_3$ then $g_1 \succ g_3$, for all $g_1, g_2, g_3 \in \mathcal{L}(\mathcal{X})$.

Strict vector ordering:

V1. $g \succ h \Leftrightarrow \lambda g \succ \lambda h$, for all $\lambda > 0$;

V2. $g_1 \succ g_2 \quad \Leftrightarrow$ $g_1 + h \succ g_2 + h$, for all $h \in \mathcal{L}(\mathcal{X})$.

Dominance $\geqslant$:
$g \geqslant h \Leftrightarrow$ $g(x) \geqslant h(x)$ for all $x \in \mathcal{X}$.

Strict dominance $>$:
$g > h \Leftrightarrow (g \geqslant h$ and $g \neq h)$.

Your beliefs about what value X assumes in $\mathcal{X}$ may lead you to (strictly) prefer one uncertain reward $g(X)$ over another $h(X)$:

$$g \succ h.$$

The binary relation $\succ$ on $\mathcal{L}(\mathcal{X})$:

- is a weak strict ordering due to basic rationality requirements;
- is a strict vector ordering due to the linearity of the utility scale;
- respects strict dominance: if $g > h$ then $g \succ h$.

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Incomparability does not imply equivalence.

We allow for PARTIAL preference!

Desirability

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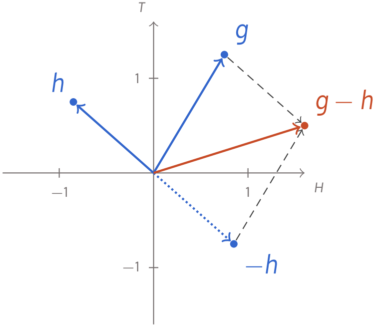

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The strict preference ordering $\succ$ is completely determined by the set of desirable gambles:

$$D = \{g \in \mathcal{L}(\mathcal{X}) \colon g \succ 0\}.$$

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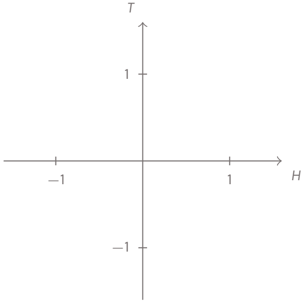

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Coherence of a set of desirable gambles D:

D1. $0 \notin D$;

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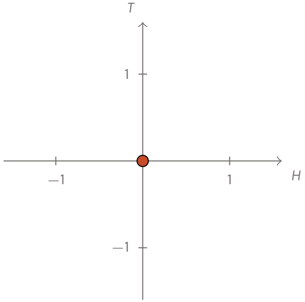

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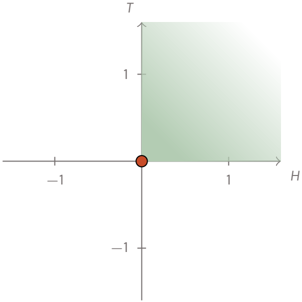

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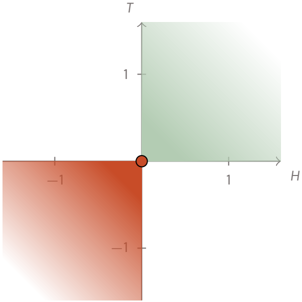

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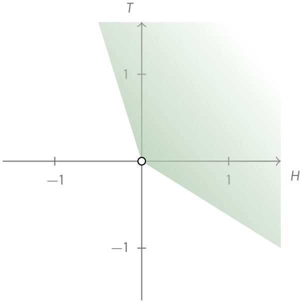

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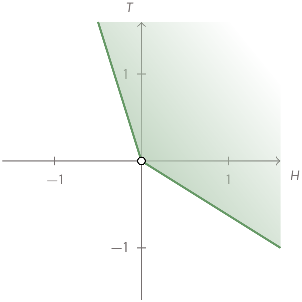

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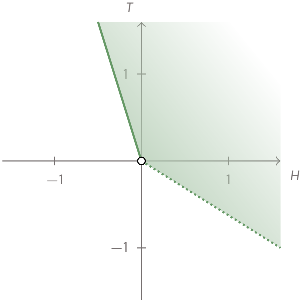

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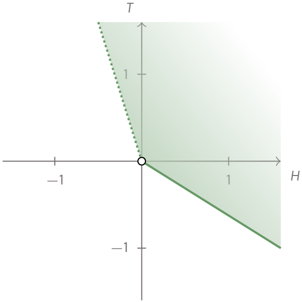

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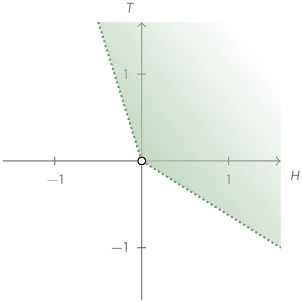

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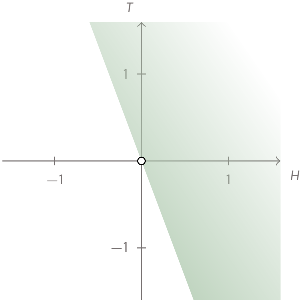

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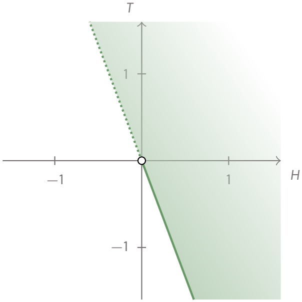

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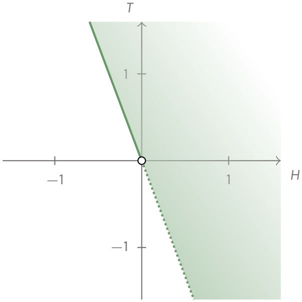

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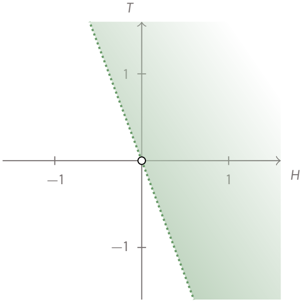

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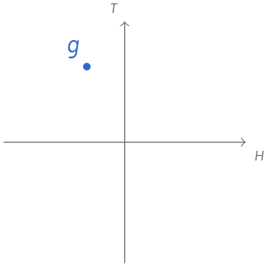
CONSERVATIVE INFERENCE

Simple desirability statements



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You state that a gamble g is desirable when You strictly prefer it to the zero gamble (status quo).

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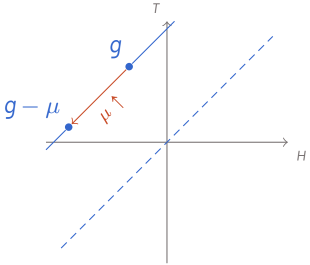

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- $\vdash_D (g-\mu)$: You are willing to buy the gamble g for a price μ;

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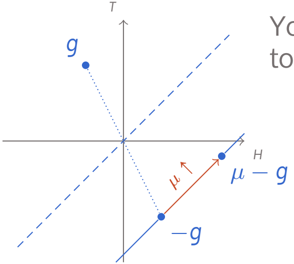

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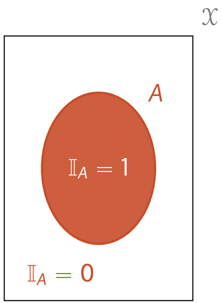

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- $\vdash_D (\mathbb{I}_A - \alpha)$: You are willing to bet on the event A at betting rate α;
- $\vdash_D (\alpha - \mathbb{I}_A)$: You are willing to bet against the event A at betting rate $1 - \alpha$

A logic of desirability statements

What are the consequences of making an assessment $\mathcal{A}$?

An assessment $\mathcal{A}$ collects (some) gambles that You find desirable:

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- we call a set $D \subseteq \mathcal{L}(\mathcal{X})$ coherent if it satisfies D1–D4.
- we collect all coherent D in the set $\overline{\mathbf{D}}$.

Conservative inference

$\overline{\mathbf{D}}$ is closed under non-empty intersections:

$$D_i \in \overline{\mathbf{D}} \text{ for all } i \in I$$
$$\Downarrow$$
$$\bigcap_{i \in I} D_i \in \overline{\mathbf{D}}$$

Conservative inference

Deductive closure operator:

$$\mathrm{cl}_{\mathbf{D}}(\mathcal{A}) := \bigcap\{D \in \overline{\mathbf{D}} \colon \mathcal{A} \subseteq D\}$$

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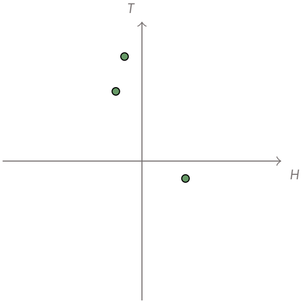

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When $\mathcal{A}$ is consistent, then

$$\mathrm{cl}_{\mathbf{D}}(\mathcal{A}) = \mathrm{posi}(\mathcal{L}_{>0} \cup \mathcal{A})$$

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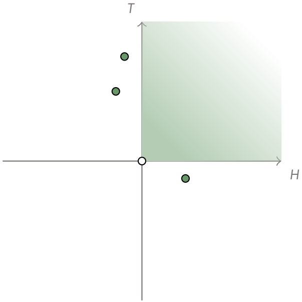

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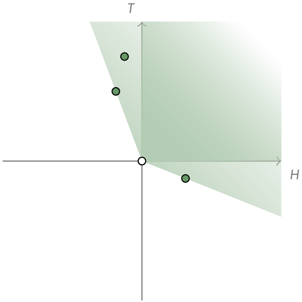

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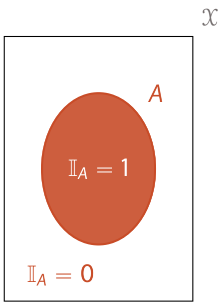

To bring in conditioning, we consider called-off gambles $\mathbb{I}_A g$

$$(\mathbb{I}_A g)(x) := \mathbb{I}_A(x)g(x) = \begin{cases} g(x) & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

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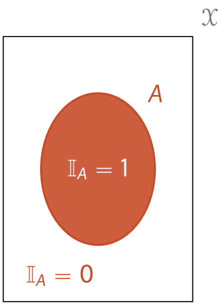

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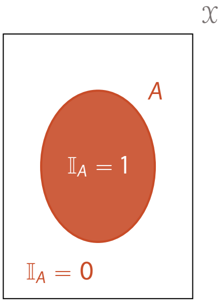
Conditional assessment in a logic of desirability statements:

g is desirable conditional on A $\longrightarrow$ $\vdash_{\mathrm{D}} \mathbb{I}_A g$ $\longrightarrow$ $\mathbb{I}_A g \in D$

Conditioning



To bring in conditioning, we consider called-off gambles $\mathbb{I}_A g$

$$(\mathbb{I}_A g)(x) := \mathbb{I}_A(x) g(x) = \begin{cases} g(x) & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Conditional assessment in a logic of desirability statements:

g is desirable conditional on A $\longrightarrow$ $\vdash_{\mathrm{D}} \mathbb{I}_A g$ $\longrightarrow$ $\mathbb{I}_A g \in D$

Conditioning operation $\cdot|A\colon \overline{\mathbf{D}} \to \overline{\mathbf{D}}\colon D \mapsto D|A$

$$g \in D|A \Leftrightarrow (g > 0 \text{ or } \mathbb{I}_A g \in D) \text{ for all } g \in \mathcal{L}(\mathcal{X})$$

A LITTLE EXERCISE

Exercise on desirability: the problem

Consider next week's football match between your favourite team *Marginal* and that unspeakable team *Zichem-Zussen-Bolder*.

You make the following statements about the outcome of this match for *Marginal*:

- A win (W) is more likely than a draw (D);
- A loss (L) is more likely than a draw;
- If no draw, then a win is more than twice as likely as a loss.

Question:

Can You consistently bet on draw at rates higher than $\frac{1}{2}$?

SUMMING UP

Advantages of the desirability approach

Working with sets of desirable gambles:

- is simple, intuitive and elegant;
- gives a geometrical flavour to probabilistic inference;
- shows that probabilistic inference is a form of 'logical inference';
- avoids problems with conditioning on sets of probability zero.

Next, we will see that:

- it allows us to derive other models, such as (conditional) lower previsions, (sets of) conditional probabilities, ...;
- it is more general and more expressive than these other models.

THE END – FOR NOW