SIPTA

SIPTA Summer School
15–19 August 2022
University of Bristol

Towards other models

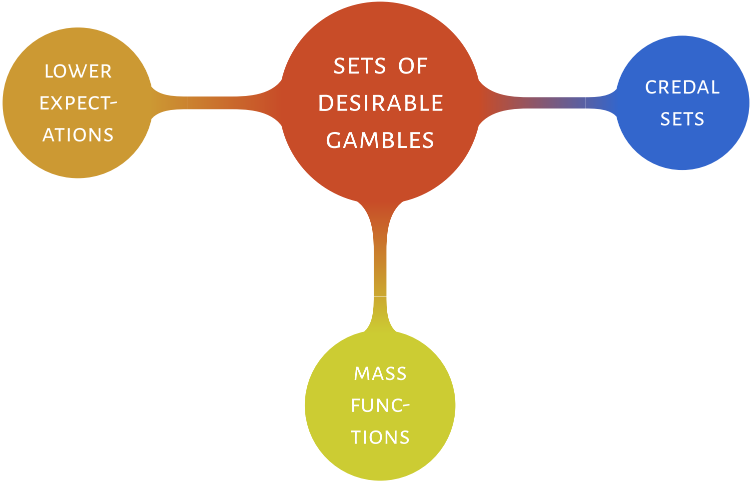
– credal sets and lower expectations

Gert de Cooman

GHENT UNIVERSITY

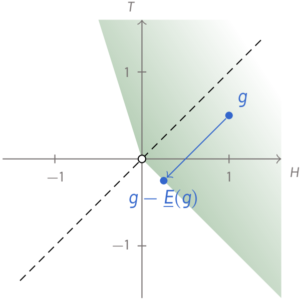
Foundations Lab
for imprecise probabilities

Overview


LOWER
EXPECT-
ATIONS
SETS OF
DESIRABLE
GAMBLES
CREDAL
SETS
MASS
FUNC-
TIONS

LOWER EXPECTATIONS

Towards lower (and upper) expectations



The lower expectation $\underline{E}(g)$ of a gamble g:

$$\underline{E}(g) := \sup\{\mu \in \mathbb{R} : g - \mu \in D\}$$

is the supremum buying price for g

The upper expectation $\overline{E}(g)$ of a gamble g:

$$\overline{E}(g) := \inf\{\mu \in \mathbb{R} : \mu - g \in D\}$$

is the infimum selling price for g

Conjugacy relationship between both:

$$\overline{E}(g) = -\underline{E}(-g) \text{ for all } g \in \mathcal{L}(\mathcal{X}).$$

Lower expectations

What are the properties of $\underline{E}$?

LE$_1$. $\min g \leqslant \underline{E}(g)$;

LE$_2$. $\underline{E}(g+h) \geqslant \underline{E}(g) + \underline{E}(h)$;

LE$_3$. $\underline{E}(\lambda g) = \lambda \underline{E}(g)$ for all $\lambda \geqslant 0$.

Exercise: prove LE$_2$ from the desirability axioms.

Lower expectations

Lower probability of event A:

$$\underline{P}(A) := \underline{E}(\mathbb{I}_A)$$

is a derived notion.

What are the properties of $\underline{E}$?

LE1. $\min g \leqslant \underline{E}(g)$;

LE2. $\underline{E}(g+h) \geqslant \underline{E}(g) + \underline{E}(h)$;

LE3. $\underline{E}(\lambda g) = \lambda \underline{E}(g)$ for all $\lambda \geqslant 0$.

Derived properties:

LE4. $\min g \leqslant \underline{E}(g) \leqslant \overline{E}(g) \leqslant \max(g)$;

LE5. $\underline{E}(g)+\underline{E}(h) \leqslant \underline{E}(g+h) \leqslant \underline{E}(g)+\overline{E}(h) \leqslant \overline{E}(g+h) \leqslant \overline{E}(g)+\overline{E}(h)$;

LE6. $\overline{E}(\lambda g) = \lambda \overline{E}(g)$ for all $\lambda \geqslant 0$.

LE7. $\underline{E}(g+\mu) = \underline{E}(g) + \mu$ and $\overline{E}(g+\mu) = \overline{E}(g) + \mu$ for all $\mu \in \mathbb{R}$.

LE8. $\underline{E}(\mu) = \overline{E}(\mu) = \mu$ for all $\mu \in \mathbb{R}$.

LE9. ...

Lower expectations

Lower probability of event A:

$$\underline{P}(A) := \underline{E}(\mathbb{I}_A)$$

is a derived notion.

Definition can be extended:

$$\mathcal{L}(\mathcal{X})$$
$$\downarrow$$
any real Banach space

'expectation' → 'prevision'

What are the properties of $\underline{E}$?

LE1. $\min g \leqslant \underline{E}(g)$;

LE2. $\underline{E}(g+h) \geqslant \underline{E}(g) + \underline{E}(h)$;

LE3. $\underline{E}(\lambda g) = \lambda \underline{E}(g)$ for all $\lambda \geqslant 0$.

Derived properties:

LE4. $\min g \leqslant \underline{E}(g) \leqslant \overline{E}(g) \leqslant \max(g)$;

LE5. $\underline{E}(g)+\underline{E}(h) \leqslant \underline{E}(g+h) \leqslant \underline{E}(g)+\overline{E}(h) \leqslant \overline{E}(g+h) \leqslant \overline{E}(g)+\overline{E}(h)$;

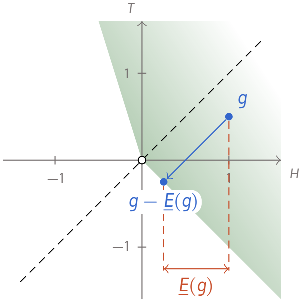
LE6. $\overline{E}(\lambda g) = \lambda \overline{E}(g)$ for all $\lambda \geqslant 0$.

LE7. $\underline{E}(g+\mu) = \underline{E}(g) + \mu$ and $\overline{E}(g+\mu) = \overline{E}(g) + \mu$ for all $\mu \in \mathbb{R}$.

LE8. $\underline{E}(\mu) = \overline{E}(\mu) = \mu$ for all $\mu \in \mathbb{R}$.

LE9. ...

Marginal gambles



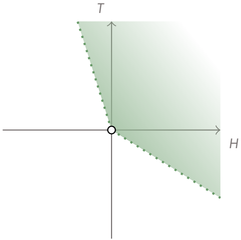
Important distinction for any gamble $g \in \mathcal{L}(\mathcal{X})$:

- ◊ $\underline{E}(g) > 0$ iff g lies in the interior of D → strictly desirable
- ◊ $\underline{E}(g) < 0$ iff g lies outside the closure of D
- ◊ $\underline{E}(g) = 0$ iff g lies on the boundary of D → marginal

If $\underline{E}(g) = 0$ we don't know if $g \in D$, unless D is Archimedean!

SETS OF DESIRABLE GAMBLES ARE MORE INFORMATIVE THAN LOWER EXPECTATIONS

Archimedean desirability models



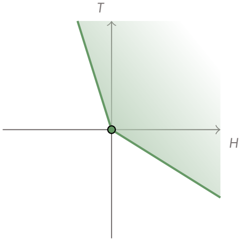
Strict desirability:

g is strictly desirable if $\big(g > 0$ or $\underline{E}(g) > 0\big)$.

and

$$g \succ h \Leftrightarrow \big(g > h \text{ or } \underline{E}(g-h) > 0\big)$$

[Sen–Walley strict preference].



Almost-desirability:

g is almost desirable if $\underline{E}(g) \geqslant 0$.

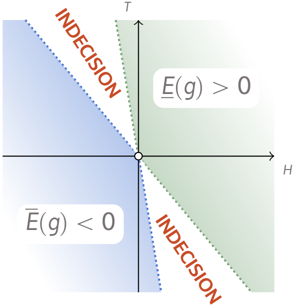
and

$$g \succeq h \Leftrightarrow \underline{E}(g-h) \geqslant 0$$

[Sen–Walley weak preference].

EXPECTATIONS AND MASS FUNCTIONS

What if there is no indecision?

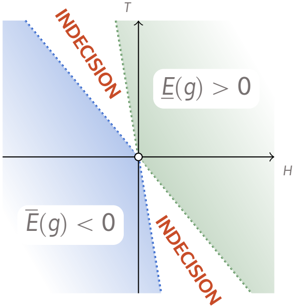


Partitioning the gamble space $\mathcal{L}(\mathcal{X})$:

- $\underline{E}(g) > 0$ means strictly better than zero;
- $\overline{E}(g) < 0$ means strictly worse than zero;
- the remaining gambles satisfy $\underline{E}(g) \leqslant 0 \leqslant \overline{E}(g)$.

But there's also gambles in the third region that are equivalent to zero!

What if there is no indecision?



Partitioning the gamble space $\mathcal{L}(\mathcal{X})$:

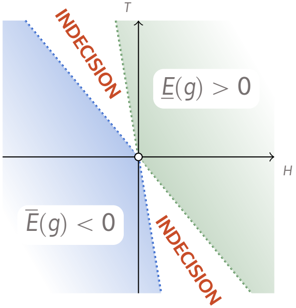
- $\underline{E}(g) > 0$ means strictly better than zero;
- $\overline{E}(g) < 0$ means strictly worse than zero;
- the remaining gambles satisfy $\underline{E}(g) \leqslant 0 \leqslant \overline{E}(g)$.

But there's also gambles in the third region that are equivalent to zero!

Such indifferent gambles **constitute a linear space**.

You are undecided about the remaining gambles.

What if there is no indecision?



Partitioning the gamble space $\mathcal{L}(\mathcal{X})$:

- $\underline{E}(g) > 0$ means strictly better than zero;
- $\overline{E}(g) < 0$ means strictly worse than zero;
- the remaining gambles satisfy $\underline{E}(g) \leqslant 0 \leqslant \overline{E}(g)$.

But there's also gambles in the third region that are equivalent to zero!

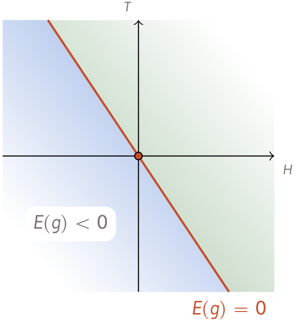
Such indifferent gambles **constitute a linear space**.

You are undecided about the remaining gambles.

Homework: prove that

$$\{g \in \mathcal{L}(\mathcal{X}) \colon \underline{E}(g) \leqslant 0 \leqslant \overline{E}(g)\} \text{ is a linear space}$$
$$\Rightarrow \underline{E}(h) = \overline{E}(h) \text{ for all gambles } h \in \mathcal{L}(\mathcal{X}).$$

Expectations



There is no indecision iff every gamble has a fair price:

$$\underline{E}(g) = \overline{E}(g) =: E(g) \text{ for all gambles } g \in \mathcal{L}(\mathcal{X}).$$

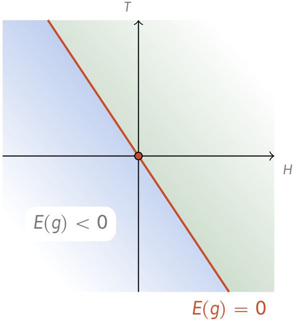
What are the properties of E?

E1. $\min g \leqslant E(g) \leqslant \max(g)$;

E2. $E(g+h) = E(g) + E(h)$;

E3. $E(\lambda g) = \lambda E(g)$ for all real λ.

Expectations



P is the set of all expectations.

There is no indecision iff every gamble has a fair price:

$$\underline{E}(g) = \overline{E}(g) =: E(g) \text{ for all gambles } g \in \mathcal{L}(\mathcal{X}).$$

What are the properties of E?

E1. $\min g \leqslant E(g) \leqslant \max(g)$;

E2. $E(g+h) = E(g) + E(h)$;

E3. $E(\lambda g) = \lambda E(g)$ for all real λ.

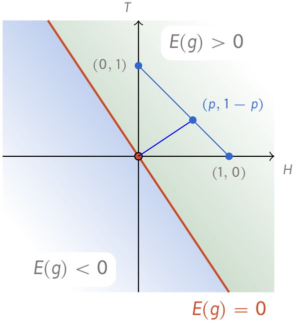
Derived properties:

E4. $E(g+\mu) = E(g) + \mu$ for all $\mu \in \mathbb{R}$.

E5. $E(\mu) = \mu$ for all $\mu \in \mathbb{R}$.

E6. ...

Mass functions



Since

$$g = \sum_{x \in \mathcal{X}} g(x)\mathbb{I}_{\{x\}},$$

it follows at once from Axioms E_2 and E_3 that

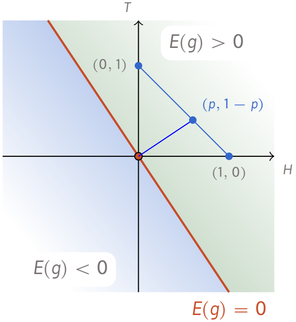
$$E(g) = E\Bigg(\sum_{x \in \mathcal{X}} g(x)\mathbb{I}_{\{x\}}\Bigg) = \sum_{x \in \mathcal{X}} g(x)E(\mathbb{I}_{\{x\}}) = \sum_{x \in \mathcal{X}} g(x)p(x) = E_p(g),$$

if we define the (probability) mass function $p\colon \mathcal{X} \to \mathbb{R}$ by

$$p(x) := E(\mathbb{I}_{\{x\}}) \text{ for all } x \in \mathcal{X}.$$

Expectations and mass functions are equivalent.

Mass functions



$\mathcal{L}(\mathcal{X})$

$\downarrow$

any real Banach space

'expectation' → 'prevision'

Since

$$g = \sum_{x \in \mathcal{X}} g(x) \mathbb{I}_{\{x\}},$$

it follows at once from Axioms E_2 and E_3 that

$$E(g) = E\Bigg(\sum_{x \in \mathcal{X}} g(x) \mathbb{I}_{\{x\}}\Bigg) = \sum_{x \in \mathcal{X}} g(x) E(\mathbb{I}_{\{x\}}) = \sum_{x \in \mathcal{X}} g(x) p(x) = E_p(g),$$

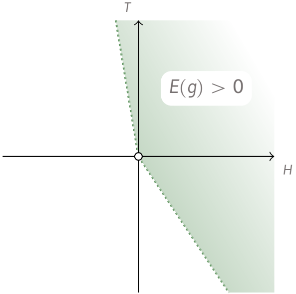
if we define the (probability) mass function $p \colon \mathcal{X} \to \mathbb{R}$ by

$$p(x) := E(\mathbb{I}_{\{x\}}) \text{ for all } x \in \mathcal{X}.$$

Expectations and mass functions are equivalent.

CREDAL SETS

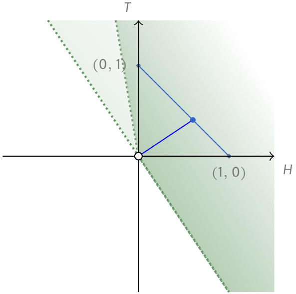
Sets of expectations, aka credal sets



Consider the set of all expectations E that dominate the lower expectation $\underline{E}$:

$$\mathbf{P}(\underline{E}) := \left\{E \in \mathbf{P} \colon (\forall h \in \mathcal{L}(\mathcal{X}))E(h) \geqslant \underline{E}(h)\right\}.$$

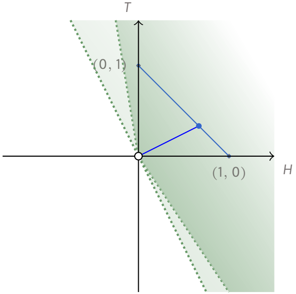
Sets of expectations, aka credal sets



Consider the set of all expectations E that dominate the lower expectation $\underline{E}$:

$$\mathbf{P}(\underline{E}) := \{E \in \mathbf{P} \colon (\forall h \in \mathcal{L}(\mathcal{X}))E(h) \geq \underline{E}(h)\}.$$

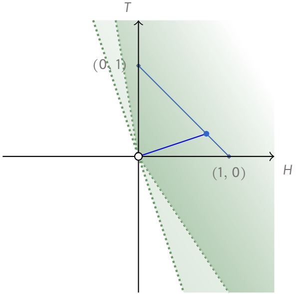
Sets of expectations, aka credal sets



Consider the set of all expectations E that dominate the lower expectation $\underline{E}$:

$$\mathbf{P}(\underline{E}) := \left\{E \in \mathbf{P} \colon (\forall h \in \mathcal{L}(\mathcal{X}))E(h) \geqslant \underline{E}(h)\right\}.$$

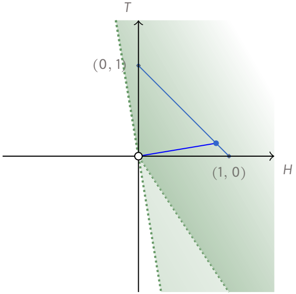
Sets of expectations, aka credal sets



Consider the set of all expectations E that dominate the lower expectation $\underline{E}$:

$$\mathbf{P}(\underline{E}) := \left\{E \in \mathbf{P} \colon (\forall h \in \mathcal{L}(\mathcal{X}))E(h) \geqslant \underline{E}(h)\right\}.$$

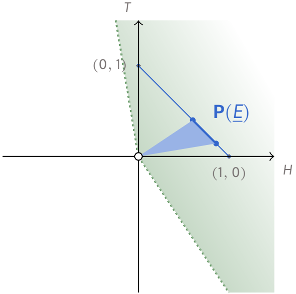
Sets of expectations, aka credal sets



Consider the set of all expectations E that dominate the lower expectation $\underline{E}$:

$$\mathbf{P}(\underline{E}) := \left\{E \in \mathbf{P} \colon (\forall h \in \mathcal{L}(\mathcal{X}))E(h) \geqslant \underline{E}(h)\right\}.$$

Sets of expectations, aka credal sets



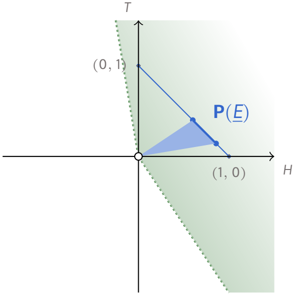
Consider the set of all expectations E that dominate the lower expectation $\underline{E}$:

$$\mathbf{P}(\underline{E}) := \left\{E \in \mathbf{P} \colon (\forall h \in \mathcal{L}(\mathcal{X}))E(h) \geqslant \underline{E}(h)\right\}.$$

$\mathbf{P}(\underline{E})$ is closed and convex, and $\underline{E}$ is the lower envelope of $\mathbf{P}(\underline{E})$:

$$\underline{E}(g) = \min\{E(g) \colon E \in \mathbf{P}(\underline{E})\}.$$

Sets of expectations, aka credal sets



Consider the set of all expectations E that dominate the lower expectation $\underline{E}$:

$$\mathbf{P}(\underline{E}) := \left\{E \in \mathbf{P} \colon (\forall h \in \mathcal{L}(\mathcal{X}))E(h) \geqslant \underline{E}(h)\right\}.$$

$\mathbf{P}(\underline{E})$ is closed and convex, and $\underline{E}$ is the lower envelope of $\mathbf{P}(\underline{E})$:

$$\underline{E}(g) = \min\{E(g) \colon E \in \mathbf{P}(\underline{E})\}.$$

DUALITY: imperfect models in terms of perfect ones

$$\{g \colon \underline{E}(g) > 0\} = \bigcap_{E \in \mathbf{P}(\underline{E})} \{g \colon E(g) > 0\}.$$

Product ordering:

$$\underline{E}(g) > 0 \Leftrightarrow \left(\forall E \in \mathbf{P}(\underline{E})\right)E(g) > 0.$$

A LITTLE EXERCISE

Exercise

Flipping a coin has two possible outcomes: heads H and tails T.

The general form of any lower expectation is given by (prove this as homework):

$$\underline{E}(g) = (1-\epsilon)\,[p(H)g(H) + p(T)g(T)] + \epsilon \min\{g(H), g(T)\},$$

where p is any probability mass function on $\{H, T\}$ and $0 \leqslant \epsilon \leqslant 1$.

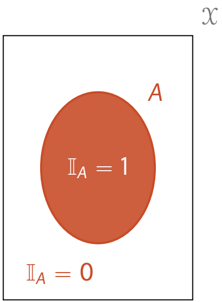
If we have no reason to prefer heads over tails, we use a symmetrical model, with $\underline{E}(\mathbb{I}_{\{H\}}) = \underline{E}(\mathbb{I}_{\{T\}})$.

Questions:

- Which lower previsions correspond to this symmetry requirement?
- Draw corresponding coherent sets of desirable gambles, and sets of mass functions.

THE CONDITIONAL VERSIONS

Conditional lower and upper expectations



Starting from a set of desirable gambles D:

$$\underline{E}(g|A) := \sup\{\alpha \in \mathbb{R} \colon g - \alpha \in D|A\}$$
$$= \sup\{\alpha \in \mathbb{R} \colon (g - \alpha)\mathbb{I}_A \in D\}$$
$$\overline{E}(g|A) := \sup\{\beta \in \mathbb{R} \colon \beta - g \in D|A\}$$
$$= \inf\{\beta \in \mathbb{R} \colon (\beta - g)\mathbb{I}_A \in D\}.$$

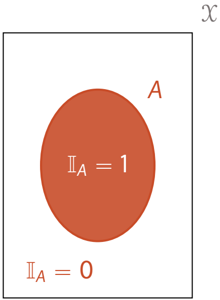
Starting from a lower expectation $\underline{E}$:

$$\underline{E}(g|A) := \sup\{\alpha \in \mathbb{R} \colon \underline{E}((g - \alpha)\mathbb{I}_A) > 0\}$$
$$\overline{E}(g|A) := \inf\{\beta \in \mathbb{R} \colon \underline{E}((\beta - g)\mathbb{I}_A) > 0\},$$

since, essentially,

$$(g - \alpha)\mathbb{I}_A \in D \Leftrightarrow \underline{E}((g - \alpha)\mathbb{I}_A) > 0.$$

Exercise: Conditional expectations and Bayes's rule



Probability of event A:

$$P(A) := E(\mathbb{I}_A)$$

is a derived notion.

Start from an expectation E and the formulas

$$\underline{E}(g|A) := \sup\{\alpha \in \mathbb{R} : E((g-\alpha)\mathbb{I}_A) > 0\}$$
$$\overline{E}(g|A) := \inf\{\alpha \in \mathbb{R} : E((\alpha-g)\mathbb{I}_A) > 0\}.$$

Show that when $P(A) > 0$ then

$$E(g|A) := \underline{E}(g|A) = \overline{E}(g|A) = \frac{E(g\mathbb{I}_A)}{P(A)}$$

and in particular, for any $B \subseteq \mathcal{X}$,

$$P(B|A) := \underline{P}(B|A) = \underline{P}(B|A) = \frac{P(B \cap A)}{P(A)}.$$

THE END – FOR NOW