SIPTA

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Non-binary preference

– and still desirability

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Foundations Lab
for imprecise probabilities

PRELIMINARIES

Rejection functions

A more general approach uses sets of gambles.

A gamble set A is a finite set of gambles:

$$A \Subset \mathcal{L}(\mathcal{X})$$

The set of all gamble sets:

$$\mathcal{Q} := \{A \colon A \Subset \mathcal{L}(\mathcal{X})\}.$$

Rejection functions

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Your rejection function is a map $R \colon \mathcal{Q} \to \mathcal{Q}$ such that

$$(\forall A \in \mathcal{Q}) R(A) \subset A$$

whose interpretation is that

> *$R(A)$ is the set of those gambles You reject from the set of gambles A that You are presented with.*

Essentially binary rejection functions

The binary aspects of a rejection function:

$$g \succ h \Leftrightarrow R(\{g, h\}) = \{h\}.$$

A rejection function R is essentially binary if

$$\begin{aligned} g \in R(A) &\Leftrightarrow (\exists h \in A)h \succ g \\ &\Leftrightarrow (\exists h \in A)R(\{g, h\}) = \{g\} \end{aligned}$$

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But we are interested in other, non-binary, rejection functions.

THE MODEL

Towards rejecting zero

R is determined if we know, for all $A \in \mathcal{Q}$ and all $g \in A$, whether

$$g \in R(A)?$$

By the linearity of the utility scale, this is equivalent to asking whether

$$0 \in R(A - g)?$$

where

$$A - g = \{h - g \colon g \in A\} = \{0\} \cup \{h - g \colon h \in A \setminus \{g\}\}.$$

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Conclusion: It is enough to know whether, for any $A \in \mathcal{Q}$,

$$0 \in R(\{0\} \cup A)?$$

Sets of desirable sets

Consider any set of gambles $A = \{g_1, g_2, \dots, g_m\} \in \mathcal{Q}$, then

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&\Leftrightarrow \vdash_{\mathrm{D}} g_1 \text{ OR } \vdash_{\mathrm{D}} g_2 \text{ OR } \dots \text{ OR } \vdash_{\mathrm{D}} g_m\,.
\end{aligned}
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Working with rejection functions means that we have to extend our desirability logic with 'OR' statements.

Sets of desirable sets

Your set of desirable sets K

$$K := \{A \in \mathcal{Q} \colon \vdash_{\mathrm{D}} A\}$$

contains the sets of gambles that You find desirable.

Consider any set of gambles $A = \{g_1, g_2, \dots, g_m\} \in \mathcal{Q}$, then

$$\begin{aligned} 0 \in R(\{0\} \cup A) &\Leftrightarrow \text{some gamble in } A \text{ is strictly preferred to } 0 \\ &\Leftrightarrow \text{some gamble in } A \text{ is desirable} \\ &\Leftrightarrow \vdash_{\mathrm{D}} g_1 \text{ OR } \vdash_{\mathrm{D}} g_2 \text{ OR } \dots \text{ OR } \vdash_{\mathrm{D}} g_m. \end{aligned}$$

Working with rejection functions means that we have to extend our desirability logic with 'OR' statements.

Alternative, but equivalent, approach:
We call a set of gambles A desirable, and write $\vdash_{\mathrm{D}} A$ if You maintain that A contains at least one desirable gamble.

The binary special case

Suppose You have a binary preference model $D \in \overline{\mathbf{D}}$.

When is a set of gambles A desirable to You?

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$$\begin{aligned} \vdash_{\mathrm{D}} A &\Leftrightarrow (\exists g \in A) \vdash_{\mathrm{D}} g \\ &\Leftrightarrow (\exists g \in A) g \in D \\ &\Leftrightarrow A \cap D \neq \emptyset. \end{aligned}$$

Your corresponding set of desirable sets is then

$$K_D := \{A \in \mathcal{Q} : A \cap D \neq \emptyset\}.$$

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In this new language, the binary models take the form K_D, $D \in \overline{\mathbf{D}}$.

THE AXIOMS

Coherence for sets of desirable sets

Consider any set of sets of gambles $W \subseteq \mathcal{Q}$.
Φ_W is the set of all selection maps

$$\phi\colon W \to \mathcal{L}(\mathcal{X})$$

satisfying

$$\phi(A) \in A \text{ for all } A \in W.$$

Moreover,

$$\phi(W) := \{\phi(A) : A \in W\}.$$

When do we call a set of desirable sets K coherent?

K$_1$. $\emptyset \notin K$; [destruction axiom]

K$_2$. if $A \in K$ then $A \setminus \mathcal{L}_{<0}(\mathcal{X}) \in K$; [production axiom]

K$_3$. if $A \in K$ and $A \subseteq B$ then $B \in K$; [production axiom]

K$_4$. if $g_\phi \in \mathrm{cl}_{\mathbf{D}}(\phi(W))$ for all $\phi \in \Phi_W$, then $\{g_\phi : \phi \in \Phi_W\} \in K$, for all $W \Subset K$. [production axiom]

Immediate consequence:

K$_5$. if $A \cap \mathcal{L}_{>0}(\mathcal{X}) \neq \emptyset$ then $A \in K$.

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Immediate consequence:

K$_5$. if $A \cap \mathcal{L}_{>0}(\mathcal{X}) \neq \emptyset$ then $A \in K$.

- we call a set of desirable sets $K \subseteq \mathcal{Q}$ coherent if it satisfies K$_1$–K$_4$.
- we collect all coherent K in the set $\overline{\mathbf{K}}$.

Conservative inference

$\overline{\mathbf{K}}$ is closed under non-empty intersections:

$$K_i \in \overline{\mathbf{K}} \text{ for all } i \in I$$
$$\Downarrow$$
$$\bigcap_{i \in I} K_i \in \overline{\mathbf{K}}$$

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Deductive closure operator:

$$\mathrm{cl}_{\mathbf{K}}(W) := \bigcap \{K \in \overline{\mathbf{K}} \colon W \subseteq K\}$$

- A set $W \subseteq \mathcal{Q}$ is deductively closed if $W = \mathrm{cl}_{\mathbf{D}}(W)$;
- $\mathbf{K} := \overline{\mathbf{K}} \cup \{\mathcal{Q}\}$ is the set of all deductively closed sets of desirable sets;
- An assessment W is consistent if $\mathrm{cl}_{\mathbf{K}}(W) \neq \mathcal{Q}$.

REPRESENTATION

Representation theorem

Proposition:
K_D is coherent iff D is.

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Representation theorem:

A set of desirable sets $K \subseteq \mathcal{Q}$ is coherent iff there is some collection $\mathcal{D} \subseteq \overline{\mathbf{D}}$ of coherent sets of desirable gambles such that

$$K = \bigcap_{D \in \mathcal{D}} K_D,$$

and in that case the largest such representing set is given by

$$\overline{\mathbf{D}}(K) := \{D \in \overline{\mathbf{D}} : K \subseteq K_D\}.$$

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More specific representation results can be found by imposing extra conditions on the sets of desirable sets K, besides coherence.

REFINEMENTS

M-Admissibility

Adding an Archimedeanity condition ...

Proposition:

K_D is coherent and Archimedean iff there is some lower expectation $\underline{E} \in \underline{\mathbf{P}}$ such that

$$D = D_{\underline{E}} := \{g \in \mathcal{L}(\mathcal{X}) : g > 0 \text{ or } \underline{E}(g) > 0\}.$$

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Representation theorem:

A set of desirable sets $K \subseteq \mathcal{Q}$ is coherent and Archimedean iff there is some collection $\underline{\mathcal{E}} \subseteq \underline{\mathbf{P}}$ of lower expectations such that

$$K = \bigcap_{\underline{E} \in \underline{\mathcal{E}}} K_{D_{\underline{E}}},$$

and in that case the largest such representing set is given by

$$\underline{\mathbf{P}}(K) := \{\underline{E} \in \underline{\mathbf{P}} : K \subseteq K_{D_{\underline{E}}}\}.$$

E-Admissibility

Adding an Archimedeanity and mixingness condition …

Proposition:

K_D is coherent, mixing and Archimedean iff there is some expectation $E \in \mathbf{P}$ such that

$$D = D_E := \{g \in \mathcal{L}(\mathcal{X}) \colon g > 0 \text{ or } E(g) > 0\}.$$

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and in that case the largest such representing set is given by

$$\mathbf{P}(K) := \{E \in \mathbf{P} \colon K \subseteq K_{D_E}\}.$$

THE END – FOR NOW