SIPTA

SIPTA Summer School
15–19 August 2022
University of Bristol

Randomness + imprecision

– what about imprecise chances?

Gert de Cooman

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Foundations Lab
for imprecise probabilities

Close collaboration between …

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… and discussions with …

Philip Dawid

Volodya Vovk

Glenn Shafer

Teddy Seidenfeld

… the beginning of an answer to questions raised by …

Terrence L. Fine (1939 – January 31, 2021)

Random sequence?

1 1 0 0 1 0 1 0 1 1 0 1 0 1 0 1 0 0 0 1 1 1 0 1 0 1 0 1 1 1 1 1 0 0 1 1 1 0 0 1 0 0 1 1 1 0 0 1 1 1 1 0 1 0 1 0 1 1 1 . . .

Random sequences and forecasts

0 1 1 0 0 1 0 1 0 …

Random sequences and forecasts

fair coin forecasts $\frac{1}{2}$ 0 $\frac{1}{2}$ 1 $\frac{1}{2}$ 1 $\frac{1}{2}$ 0 $\frac{1}{2}$ 0 $\frac{1}{2}$ 1 $\frac{1}{2}$ 0 $\frac{1}{2}$ 1 $\frac{1}{2}$ 0 ...

Random sequences and forecasts

precise forecasts p_1 0 p_2 1 p_3 1 p_4 0 p_5 0 p_6 1 p_7 0 p_8 1 p_9 0 ...

$p_k \in [0, 1]$

Random sequences and forecasts

interval forecasts I_1 0 I_2 1 I_3 1 I_4 0 I_5 0 I_6 1 I_7 0 I_8 1 I_9 0 ...

$I_k \in \mathcal{J}$

Randomness and calibration

Randomness is about:

outcome sequence and forecast sequence

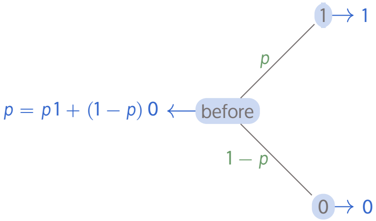
'going together well'.

→ which outcome sequences are random for a given forecast sequence?

→ which forecast sequences make a given outcome sequence random?

FORECASTS: A SINGLE OUTCOME

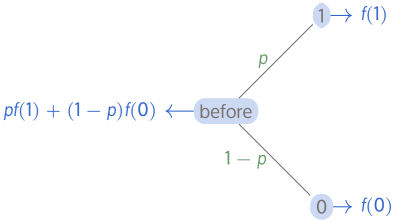
Forecast p for an X in $\mathcal{X}$

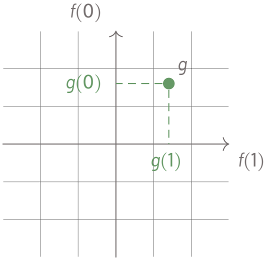


Forecast p for an X in $\mathcal{X}$

Forecasting gambles f on X:

$$f(X) \to pf(1) + (1-p)f(0)$$





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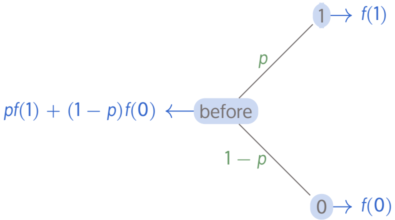
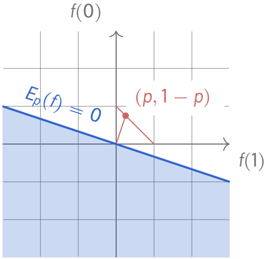




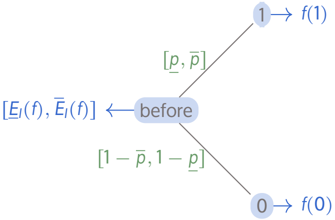
Expectation

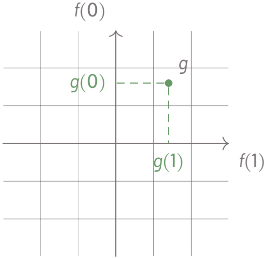
$$E_p(f) := pf(1) + (1-p)f(0) \leqslant 0$$

Interval forecast $I = [\underline{p}, \overline{p}]$ for an X in $\mathcal{X}$

Forecasting gambles f on X:

$f(X) \rightarrow$ anything between $E_{\underline{p}}(f)$ and $E_{\overline{p}}(f)$





Interval forecast $I = [\underline{p}, \overline{p}]$ for an X in $\mathcal{X}$

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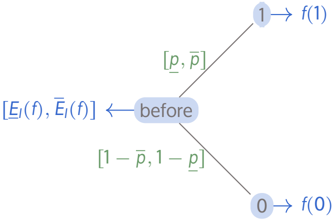
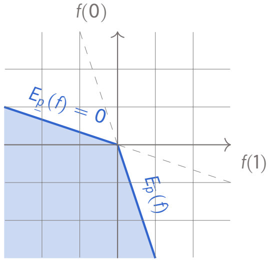




Upper expectation

$$\overline{E}_I(f) := \max\{E_{\underline{p}}(f), E_{\overline{p}}(f)\} \leqslant 0$$

FORECASTS: MANY OUTCOMES

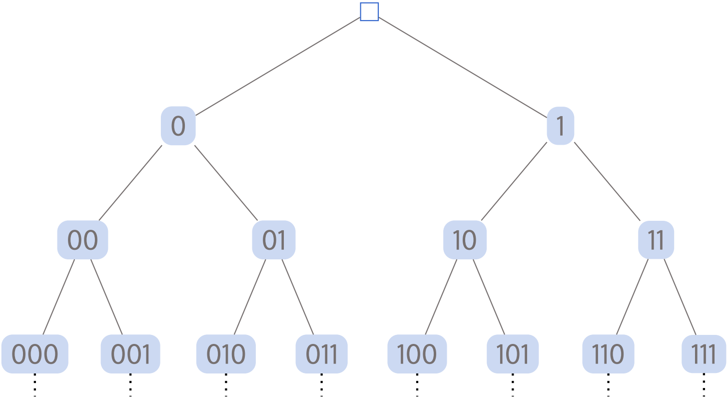
Many precise forecasts: event tree

A situation is a node in the event tree, or a finite sequence of zeroes and ones:

$$s = (x_1, \dots, x_n) \in \mathbb{S}.$$

A path is an infinite sequence of zeroes and ones:

$$\omega = (x_1, \dots, x_n, \dots) \in \Omega.$$

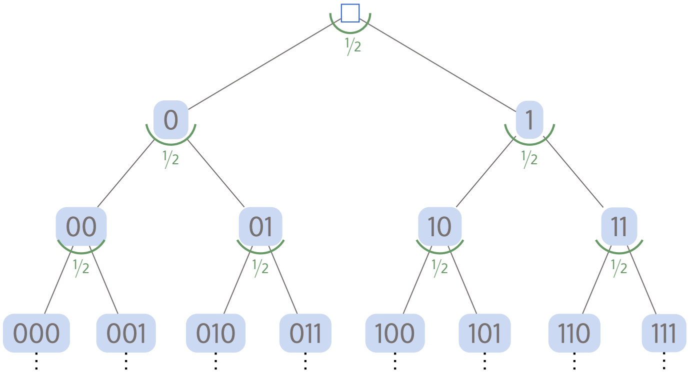


Many precise forecasts: the fair-coin tree

In a fair-coin tree, we associate a precise forecast $\frac{1}{2}$ with each situation $s \in \mathbb{S}$.

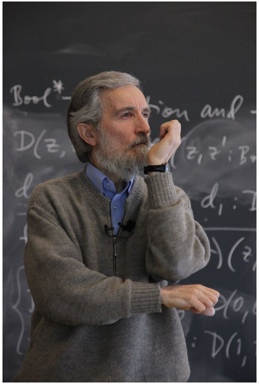
Forecasting system

$$\varphi_{1/2} \colon \mathbb{S} \to [0,1] \colon s \mapsto \varphi(s) := \frac{1}{2}$$



Many forecasts: randomness with test

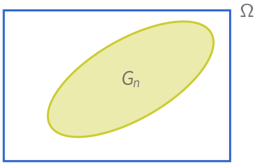
Per Martin-Löf



Going global

The local models lead to a global probability measure $P^{\varphi_{1/2}}$ on the set of all paths $\Omega = [0, 1]$.

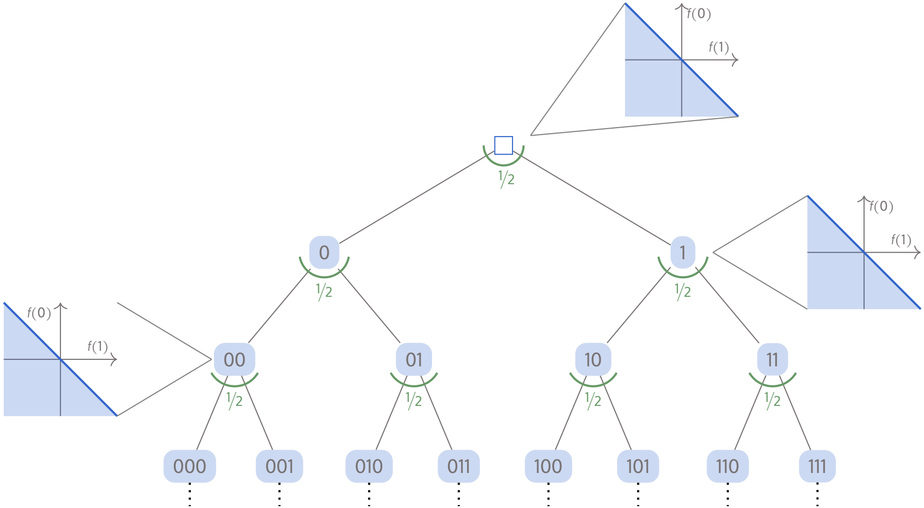
A randomness test corresponds to a (computable) sequence of (effectively) open sets G_n, with $P^{\varphi_{1/2}}(G_n) \leqslant 2^{-n}$.



A path ω succeeds the randomness test if $\omega \notin \bigcap_n G_n$.

A path ω is Martin-Löf test random if it succeeds all randomness tests.

Many forecasts: randomness with (super)martingales



Many forecasts: randomness with (super)martingales

Jean Ville

Going local

A supermartingale is a capital process where a Skeptic selects, in each situation s, one of the gambles made available by the Forecaster's forecast $\varphi_{1/2}(s)$, and where the capital is accumulated as we move through the tree.

A betting strategy is a way of selecting an available gamble f_s in each situation s.

A path ω is Martin-Löf random if there is no (lower semicomputable) strategy for which the accumulated capital remains non-negative everywhere and becomes unbounded on ω.

The local and global approaches are equivalent!

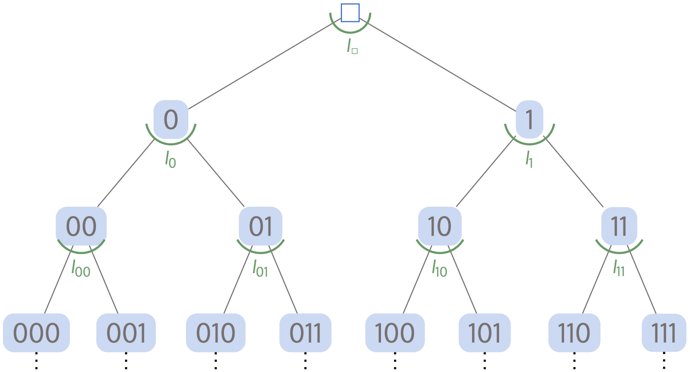
Claus Peter Schnorr

Leonid Anatolievich Levin

RANDOMNESS
WITH INTERVAL FORECASTS?

Many interval forecasts: the imprecise probability tree

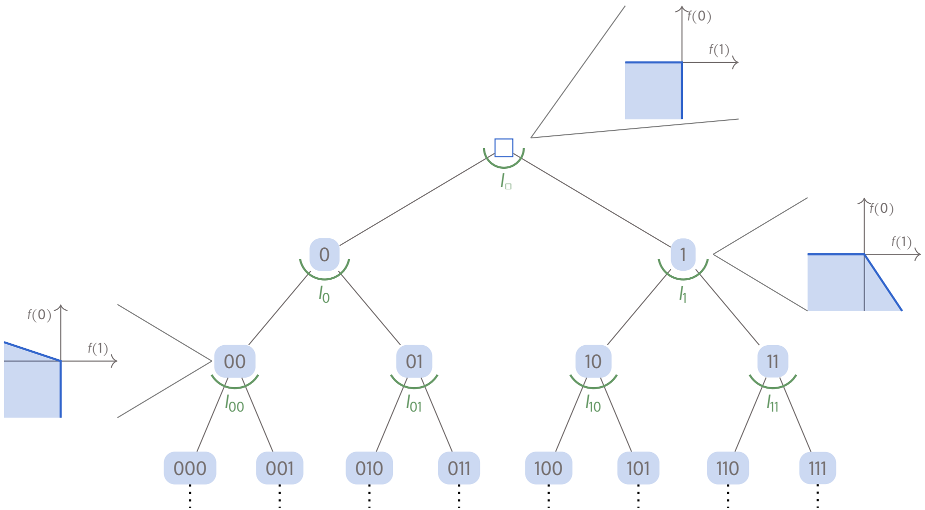
In an imprecise probability tree, we associate an interval forecast I_s with each situation $s \in \mathbb{S}$.



Forecasting system

$$\varphi\colon \mathbb{S} \to [0,1]\colon s \mapsto \varphi(s) := I_s$$

Interval forecasts: randomness with (super)martingales



Interval forecasts: randomness with (super)martingales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A supermartingale is a capital process where a Sceptic selects, in each situation s, one of the gambles made available by the Forecaster's forecast $\varphi(s)$, and where the capital is accumulated as we move through the tree.

A betting strategy is a way of selecting an available gamble f_s in each situation s.

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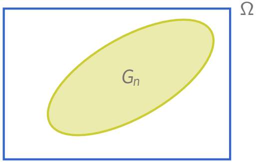
The local and global approaches are equivalent!

Going global

The local models lead to a global upper expectation $\overline{E}^{\varphi}$ on the set of all paths $\Omega = [0,1]$.

A randomness test corresponds to a (computable) sequence of (effectively) open sets G_n, with $\overline{P}^{\varphi}(G_n) \leqslant 2^{-n}$.



A path ω succeeds the randomness test if $\omega \notin \bigcap_n G_n$.

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CONSISTENCY RESULTS

Consistency

Forecaster believes he's well-calibrated

For any forecasting system $\varphi\colon \mathbb{S} \to \mathcal{I}$, almost all paths are Martin-Löf random for φ in the imprecise probability tree that corresponds to φ.

Consistency

Let

$R := \{\omega \in \Omega \colon \omega \text{ random for } \varphi\}$

then

$$\overline{P}^{\varphi}(\mathrm{co}\,R) = 0.$$

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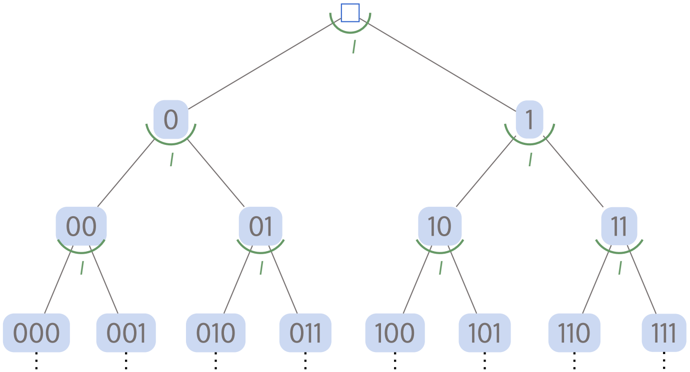
In a measure-theoretic sense, random paths are legion.

Corollary

For any forecasting system $\varphi \colon \mathbb{S} \to \mathcal{I}$, there is at least one path that is Martin-Löf random for φ.

Constant interval forecasts

Stationary forecasting system $\varphi_I(s) := I$ for all $s \in \mathbb{S}$.



Church randomness

Selection process

$$S\colon \mathbb{S} \to \mathcal{X}$$

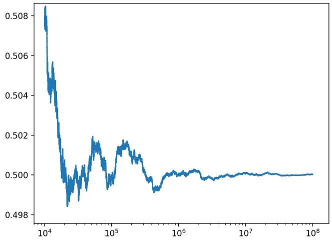
If $S(\omega_{1:k}) = 1$ then outcome ω_{k+1} is selected in the sum.

If $S(\omega_{1:k}) = 0$ then outcome ω_{k+1} is not selected in the sum.

Classical Church randomness

Any path ω is Church random if for any recursive selection process $S\colon \mathbb{S} \to \mathcal{X}$ such that $\sum_{k=0}^{n} S(\omega_{1:k}) \to \infty$:

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\omega_{k+1}}{\sum_{k=0}^{n-1} S(\omega_{1:k})} = \frac{1}{2}.$$



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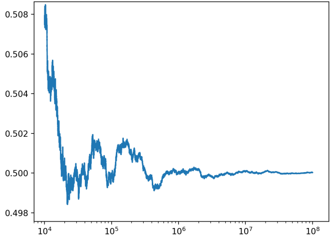
Martin-Löf randomness

$\Rightarrow$

Church randomness



Church randomness

Theorem

Consider any path ω and any constant interval forecast I that makes ω Martin-Löf random.

Then for any recursive selection process $S\colon \mathbb{S} \to \mathcal{X}$ such that $\sum_{k=0}^{n} S(\omega_{1:k}) \to \infty$:

$$\min I \leqslant \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\omega_{k+1}}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leqslant \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\omega_{k+1}}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leqslant \max I.$$

RANDOMNESS IS INHERENTLY IMPRECISE

A few simple examples

Consider any p in $[0,1]$, and the precise forecasting system φ_p defined by

$$\varphi_p(x_1,\dots,x_n) := p \quad \text{for all } (x_1,\dots,x_n) \in \mathbb{S}.$$

Proposition

For any path ω that is Martin-Löf random for φ_p, and for all $I \in \mathcal{I}$:

$$I \in \mathcal{I}_{\mathrm{ML}}(\omega) \Leftrightarrow p \in I.$$

A few simple examples

Proposition

For any recursive path ω that has infinitely many zeroes and ones, and for all $I \in \mathcal{I}$:

$$I \in \mathcal{I}_{\mathrm{ML}}(\omega) \Leftrightarrow I = [0,1].$$

A few simple examples

Consider any p and q in $[0,1]$ with $p \leqslant q$, and the forecasting system $\varphi_{p,q}$ defined by

$$\varphi_{p,q}(x_1,\dots,x_n) := \begin{cases} p & \text{if } n \text{ is odd} \\ q & \text{if } n \text{ is even} \end{cases} \quad \text{for all } (x_1,\dots,x_n) \in \mathbb{S}.$$

Proposition

For any path ω that is Martin-Löf random for $\varphi_{p,q}$, and for all $I \in \mathcal{I}$:

$$I \in \mathcal{I}_{\mathrm{ML}}(\omega) \Leftrightarrow [p,q] \subseteq I.$$

A few simple examples

Consider the precise forecasting system $\varphi_{\sim 1/2}$ defined by

$$\varphi_{\sim 1/2}(x_1, \dots, x_n) := \frac{1}{2} + (-1)^n \sqrt{\frac{8}{n+33}} \text{ for all } (x_1, \dots, x_n) \in \mathbb{S}.$$

Proposition

For any path ω that is Martin-Löf random for $\varphi_{\sim 1/2}$, and for all $I \in \mathcal{I}$:

$$I \in \mathcal{I}_{\mathrm{ML}}(\omega) \Leftrightarrow \min I < \frac{1}{2} \text{ and } \max I > \frac{1}{2}.$$

Randomness is inherently imprecise

Theorem

For any stationary interval forecast I there is some path $\omega \in \Omega$ that is Martin-Löf random for φ_I, but never Martin-Löf random for any computable forecasting system φ with smaller imprecision:

$$\sup_{s \in \mathbb{S}} \big[\max \varphi(s) - \min \varphi(s)\big] < \max I - \min I.$$

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For any stationary interval forecast I there is some path $\omega \in \Omega$ that is Martin-Löf random for φ_I, but never Martin-Löf random for any computable forecasting system φ with smaller imprecision:

$$\sup_{s \in \mathbb{S}} \left[\max \varphi(s) - \min \varphi(s) \right] < \max I - \min I.$$

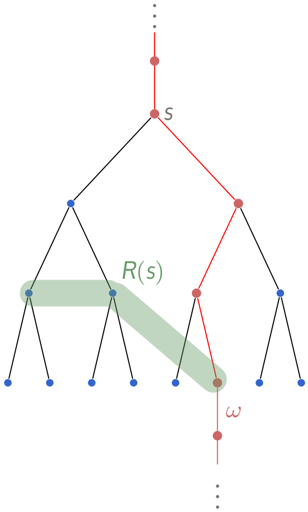
Theorem

For any non-vanishing interval forecast I there is some precise (and then necessarily) non-computable forecasting system φ such that for any path $\omega \in \Omega$:

ω is Martin-Löf random for $\varphi_I \Leftrightarrow \omega$ is Martin-Löf random for φ

RANDOM PATHS ARE RARE

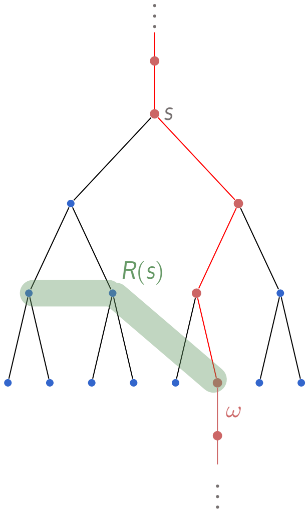
Lawful paths



Lawfulness

A path $\omega \in \Omega$ is called lawful if there is some algorithm that, given as input any situation s on the path ω, outputs a restrictive finite set $R(s)$ of extensions of s, at least one of which is also on the path.

Lawful paths



Lawfulness

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Theorem (Muchnik et al., 1998)

Any set containing only lawful paths is meagre—a countable union of nowhere dense sets.

Random paths are topologically rare

Proposition

If a path $\omega \in \Omega$ is not lawful then

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\omega_k = 0 \text{ and } \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\omega_k = 1.$$

Theorem

Let I be any interval forecast that is strictly included in $[0,1]$, so $\min I > 0$ or $\max I < 1$.

Then the set of all paths that are Martin-Löf random for the stationary forecasting system φ_I is meagre.

THE END